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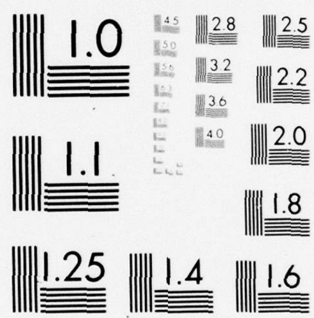

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NAVAL POSTGRADUATE SCHOOL

Monterey, California

August 1977

THE SECOND-ORDER SOLUTION FOR WAVE

INTERACTION WITH A SUBMERGED CYLINDER

by

C. J. Garrison

and

G. W. Smith

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NAVAL POSTGRADUATE SCHOOL
Monterey, California

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ABSTRACT

The second-order solution of the problem of the interaction of a train of regular waves with a completely submerged, horizontal circular cylinder in finite depth water is presented for the two-dimensional case. The incident wave is developed as a second-order Stokes' wave by use of a perturbation method and the solution of both the first-order and second-order scattering potentials is obtained numerically using the Green's function approach. The hydrodynamic pressure and resulting first-order and second-order force coefficients are determined numerically and presented for various values of water depth, cylinder depth of submergence, and wave length.

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SYMBOL INDEX

Symbol	Description
$\overline{a}$	characteristic length, cylinder radius
a	dimensionless wavelength parameter, $a = 2\pi\overline{a}/\overline{L}$
b	dimensionless constant, $H = \varepsilon b$
C_i	dimensionless wave force coefficient in the i^{th} direction
d	dimensionless relative cylinder depth
$\overline{d}$	cylinder depth
dC_1	differential dimensionless arc length on the cylinder surface
dC_2	differential dimensionless length on the free surface
f_1	first-order source strength function
f_2	second-order source strength function
f^*	free surface particular solution portion of the second-order problem source strength function
$F(\ )$	function of the quantities in parentheses
F_{1i}	dimensionless first-order wave force coefficient in the i^{th} direction
F_{2i}	dimensionless second-order periodic wave force coefficient in the i^{th} direction
F_{2i}^{SS}	dimensionless second-order steady-state wave force coefficient in the i^{th} direction
g	acceleration of gravity
G	Green's function
G^*	modified Green's function for the particular solution portion of the second-order problem

Symbol	Description
h	dimensionless relative mean water depth
$\overline{h}$	mean water depth
h_i	first-order non-homogeneous boundary condition function at the i^{th} nodal point
H	dimensionless relative wave height, $H = \overline{H}/2\overline{a}$
$\overline{H}$	elevation difference between the wave crest and trough
i	complex plane portion, $i = (-1)^{\frac{1}{2}}$
k	wave number, $k = 2\pi/\overline{L}$
k_i	second-order non-homogeneous boundary condition function at the i^{th} nodal point
K	dimensionless Bernoulli constant
$\overline{K}$	Bernoulli constant
K_2	second-order dimensionless Bernoulli constant
$K(\ )$	function of the quantities in parentheses
L	dimensionless wave length
$\overline{L}$	wave length
m	number of cylinder surface increments and nodal points
n	number of free surface increments and nodal points
$\hat{n}$	unit normal vector on the cylinder surface in the outward direction
n_x, n_y	spacial component unit normals in the horizontal and vertical directions, respectively
O	order of
p	dimensionless pressure coefficient
P	Pressure
PV	principal value

Symbol	Description
$\hat{q}$	fluid velocity vector
r	dimensionless linear distance
r'	dimensionless image linear distance
R_e	real part of a complex number
S_1	cylinder surface
S_2	free surface
t	dimensionless time, $t = \sigma\bar{t}$
$\bar{t}$	time
u_1	first-order complex potential function
u_2	second-order periodic complex potential function
$\tilde{u}_2$	second-order time independent complex potential function
U	non-periodic function
x,y	dimensionless spacial variables in the horizontal and vertical directions, respectively
$\bar{x},\bar{y}$	spacial variables in the horizontal and vertical directions, respectively
α	complex matrix
β	complex matrix
Δ	increment
∇	vector operator
δ	phase shift angle
ε	perturbation parameter
ξ,η	dimensionless spacial variables corresponding in direction to x and y, respectively
η	dimensionless free surface elevation

Symbol	Description
$\bar{\eta}$	free surface elevation
θ	plane polar angle
λ	half surface interval
μ	dummy of integration variable
μ_k	positive roots of $\mu_k \tan(\mu_k h) - \nu = 0$
ν	dimensionless wave frequency
π	constant, 3.14159
ρ	fluid density
Σ	summation of terms
σ	wave frequency
ϕ	dimensionless velocity potential
Φ	velocity potential

Subscripts

i	direction of a force or pressure component or nodal point location, depending on context
j	nodal point location
n	normal partial derivative
t	time derivative
$x, \bar{x}$	spacial partial derivative in the horizontal direction
$y, \bar{y}$	spacial partial derivative in the vertical direction
1	first-order component or x spacial direction, depending on context
2	second-order component or y spacial direction, depending on context

Symbol	Description
Superscripts	
I	incident wave component
o	homogeneous solution portion of second-order scattering problem
S	scattering wave component
*	particular solution portion of second-order scattering problem

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I. INTRODUCTION

The solution to the two-dimensional linear wave/structure interaction problem has been well-studied for both the finite and infinite depth cases. The fluid motion resulting from the interaction of a train of regular waves with a submerged horizontal circular cylinder in infinite depth water was first studied by Dean [Ref. 1]. Ursell [Ref. 7] later studied the problem anew, placing the solution on a rigorous basis. More recently, Ogilvie [Ref. 6] reconsidered the problem. He computed the first-order oscillatory forces and second-order steady-state forces for the following cases: (a) the cylinder held fixed, (b) the cylinder forced to oscillate sinusoidally in still water, and (c) the neutrally buoyant cylinder allowed to respond to wave motion.

It can easily be shown that the steady-state part of the second-order forces can be computed from the first-order potential alone. Accordingly, Ogilvie did not obtain a complete second-order solution; the steady-state forces arise from the first-order velocity-squared terms in Bernoulli's equation. In addition to the steady-state forces, second-order oscillatory forces also exist which have a frequency twice that of the first-order forces and represent the subject of the present work.

This report reconsiders again the submerged horizontal cylinder problem, extending the analysis to include water

of finite depth. The cylinder is considered to be completely submerged and held fixed in a train of regular waves. The objective is to determine the complete second-order solution and, accordingly, determine the oscillatory second-order forces as well as the steady-state second-order forces.

The problem is treated as a regular perturbation problem in the small parameter, incident wave height/cylinder radius. Proceeding in this way the incident wave appears as a second-order Stoke's wave. The boundary-value problems for both the first-order and second-order potentials are established and the solutions to both are obtained by use of the Green's function method.

II. THEORETICAL DEVELOPMENT

A. FORMULATION OF THE PROBLEM

A systematic representation of the problem considered is illustrated in Fig. (1). A rigid right cylinder of radius $\bar{a}$, submerged to a depth $\bar{d}$ in water of depth $\bar{h}$, is fixed in a train of regular waves propagating in the positive $\bar{x}$ direction. The basic problem is that of calculating the hydrodynamic pressure on the immersed cylinder and the resulting force correct to the second-order in the wave height of the incident wave.

Assuming the fluid to be irrotational, a velocity potential, Φ, may be defined as:

$$\hat{q} = \bar{\nabla}\Phi(\bar{x},\bar{y},\bar{t}) \tag{1}$$

where $\hat{q}$ denotes the fluid velocity vector. (The barred quantities denote dimensional quantities.) Moreover, assuming the fluid to be incompressible, and homogeneous, the velocity potential must satisfy the Laplace equation,

$$\bar{\nabla}^2\Phi(\bar{x},\bar{y},\bar{t}) = 0 \tag{2}$$

within the fluid region.

In addition to the differential equation, Eq. (2), Φ must satisfy certain boundary conditions. In specific, these are:

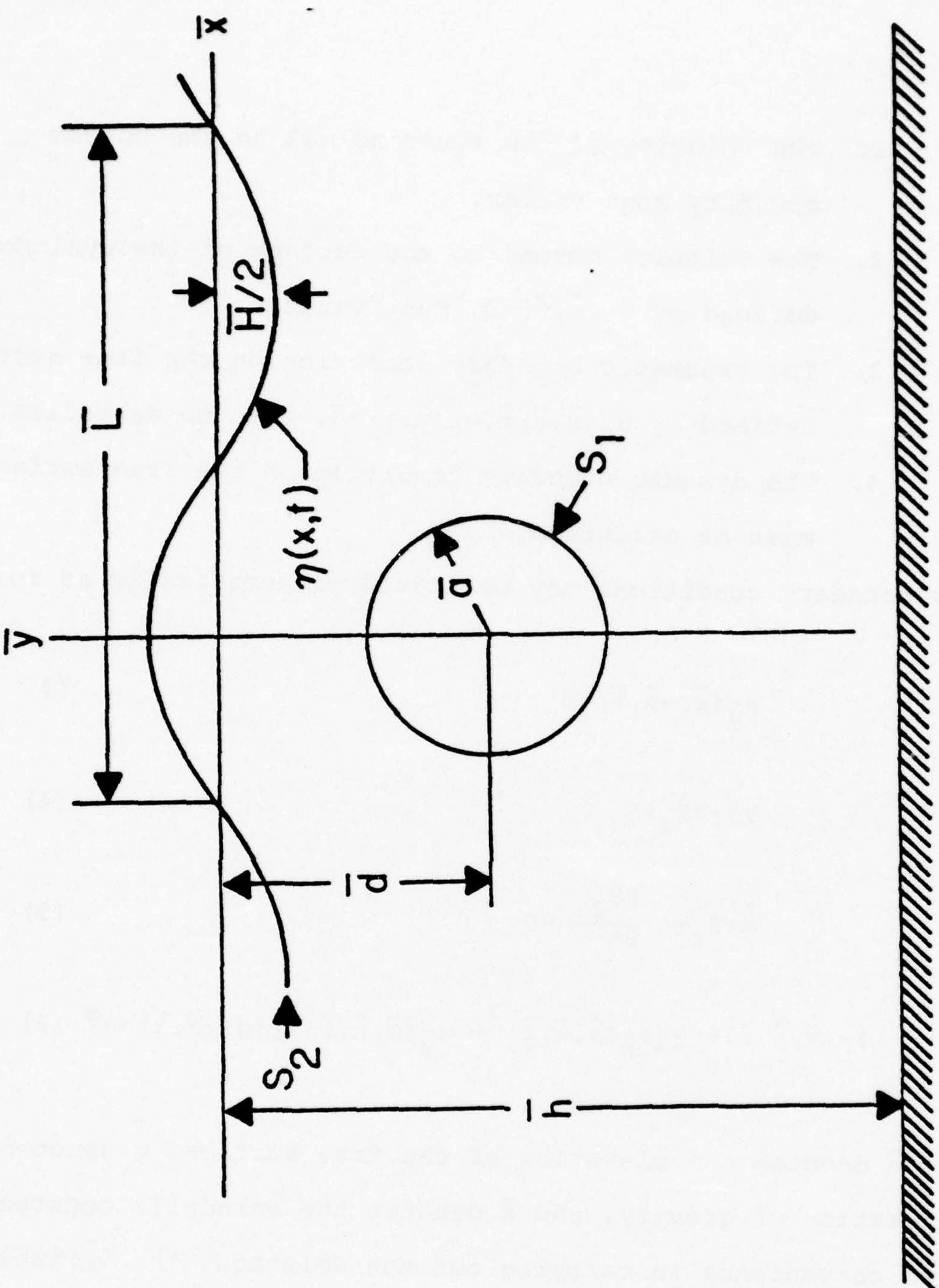

Figure 1. DEFINITION SKETCH

1. The velocity of the fluid normal to the bottom boundary must vanish.
2. The velocity normal to the surface of the cylinder, defined by $S_1(\bar{x},\bar{y})=0$, must vanish.
3. The kinematic boundary condition on the free surface, defined by $S_2(\bar{x},\bar{y},\bar{t})=\bar{\eta}(\bar{x},\bar{t})=0$, must be satisfied.
4. The dynamic boundary condition on the free surface must be satisfied.

These boundary conditions may be stated mathematically as follows:

$$\Phi_{\bar{y}}(\bar{x},-\bar{h},\bar{t})=0 \tag{3}$$

$$\nabla\Phi\cdot\nabla S_1=0 \tag{4}$$

$$\hat{q}\cdot S_2+\frac{\partial S_2}{\partial t}=0 \tag{5}$$

$$\Phi_{\bar{t}}(\bar{x},\bar{\eta},\bar{t})+\tfrac{1}{2}\left[\Phi_{\bar{x}}(\bar{x},\bar{\eta},\bar{t})^2+\Phi_{\bar{y}}(\bar{x},\bar{\eta},\bar{t})^2\right]+g\bar{\eta}(\bar{x},\bar{t})=g\bar{K} \tag{6}$$

where $\bar{\eta}$ denotes the elevation of the free surface, g denotes the acceleration of gravity, and $\bar{K}$ denotes the Bernoulli constant.

For convenience in carrying out the solution, the variables are next made dimensionless using the cylinder radius, and the wave frequency, σ, as follows:

$$
\left.\begin{array}{llll}
x = \bar{x}/\bar{a} & y = \bar{y}/\bar{a} & d = \bar{d}/\bar{a} & h = \bar{h}/\bar{a} \\
H = \bar{H}/2\bar{a} & K = \bar{K}/\bar{a} & \nu = \sigma^2\bar{a}/g & t = \sigma\bar{t} \\
\phi = \sigma\Phi/g\bar{a} & \eta = \bar{\eta}/\bar{a} & &
\end{array}\right] \quad (7)
$$

H denotes the dimensionless wave height, K denotes the dimensionless Bernoulli constant, and t denotes the dimensionless time.

Utilizing the parameters defined in Eq. (7), the boundary value problem defined in Eqs. (2-6) may be **rewritten** concisely in dimensionless form as:

$$\nabla^2\phi(x,y,t) = 0 \qquad \text{in the fluid region} \qquad (8)$$

$$\phi_y(x,-h,t) = 0 \qquad (9)$$

$$\phi_n(x,y,t) = 0 \qquad \text{on } S_1(x,y) = 0 \qquad (10)$$

$$\phi_x(x,\eta,t)\eta_x(x,t) - \phi_y(x,\eta,t) + \nu\eta_t(x,t) = 0 \qquad (11)$$

$$\phi_t(x,\eta,t) + \frac{1}{2}[\phi_x(x,\eta,t)^2 + \phi_y(x,\eta,t)^2] + \eta(x,t) = K \qquad (12)$$

B. PERTURBATION EXPANSION

Expanding ϕ and η in terms of a perturbation parameter, ε, provides a means for converting the nonlinear boundary-value

problem into a series of linear problems. The solution to each linear problem may be tractable whereas the nonlinear problem is unsolvable.

Since ϕ, η, and K are functions of the small parameter, ε, they may be written in the power series:

$$\phi(x,y,t;\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(x,y,t) \tag{13}$$

$$\eta(x,t;\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n \eta_n(x,t) \tag{14}$$

and

$$K = \sum_{n=2}^{\infty} \varepsilon^n K_n \tag{15}$$

In Eqs. (11) and (12), ϕ contains η, and, therefore, ε, implicitly. This can be converted to an explicit form in ε by use of the Taylor series expansion:

$$\phi(x,\eta,t) = \sum_{m=0}^{\infty} \frac{\eta(x,t;\varepsilon)^m}{m!} \left[\frac{\partial^m \phi(x,y,t)}{\partial y^m}\right]_{y=0} \tag{16}$$

The perturbation parameter, ε, is related to the wave height in an, as yet, unknown manner.

Equations (13-16) as well as appropriate derivatives similar to Eqs. (13-16) may now be substituted into the boundary-value problem given in Eqs. (8-12) to obtain a series of linear boundary-value problems for ϕ_1, ϕ_2, etc.

That is, upon substitution and equating coefficients of like powers of ε, the following problems for the first two terms in the expansion for ϕ can be precipitated:

First-order boundary-value problem (ε):

$$\nabla^2\phi_1(x,y,t) = 0 \tag{17}$$

$$\phi_{1y}(x,-h,t) = 0 \tag{18}$$

$$\phi_{1n}(x,y,t) = 0 \qquad \text{on } S_1(x,y) = 0 \tag{19}$$

$$\phi_{1y}(x,0,t) - \nu\eta_{1t}(x,t) = 0 \tag{20}$$

$$\eta_1(x,t) + \phi_{1t}(x,0,t) = 0 \tag{21}$$

Second-order boundary-value problem (ε^2):

$$\nabla^2\phi_2(x,y,t) = 0 \tag{22}$$

$$\phi_{2y}(x,-h,t) = 0 \tag{23}$$

$$\phi_{2n}(x,y,t) = 0 \qquad \text{on } S_1(x,y) = 0 \tag{24}$$

$$\begin{aligned}\phi_{2y}(x,0,t) - \nu\eta_{2t}(x,t) &= \\ -\eta_1(x,t)\phi_{1yy}(x,0,t) &+ \phi_{1x}(x,0,t)\eta_{1x}(x,t)\end{aligned} \tag{25}$$

$$\eta_2(x,t) + \phi_{2t}(x,0,t) = -\eta_1(x,t)\phi_{1yt}(x,0,t) -$$

$$\eta_{1t}(x,t)\phi_{1y}(x,0,t) - \frac{1}{2\nu}[\phi_{1x}(x,0,t)^2 + \phi_{1y}(x,0,t)^2] + K_2 \tag{26}$$

There is a striking similarity between the first-order and second-order problems; only the right hand side of the free surface boundary conditions differ. In the first-order problem the free surface boundary conditions are homogeneous whereas the second-order free surface boundary conditions are non-homogeneous, the right hand side being functions of the first-order velocity potential, first-order free surface elevation, and their derivatives.

The first-order potential may be represented by a function which is periodic with the fundamental frequency. Accordingly, the complex potential, $u_1(x,y)$, may be expressed as:

$$\phi_1(x,y,t) = abR_e\,[iu_1(x,y)e^{-it}] \tag{27}$$

where R_e denote the real part, $a = 2\pi\bar{a}/\bar{L}$, $\bar{L}$ denoting the wave length, and b is an unknown real constant.

C. THE FIRST-ORDER BOUNDARY VALUE PROBLEM

Since the first-order boundary-value problem is linear, the total first-order potential may be expressed as the sum,

$$\phi_1 = \phi_1^I + \phi_1^S \tag{28}$$

where ϕ_1^I denotes the incident wave potential and ϕ_1^S denotes the scatter potential which is due to the presence of the cylinder. The space dependent part of the complex potentials are expressed, in view of Eqs. (27) and (28) as,

$$u_1 = u_1^I + u_1^S \tag{29}$$

The potential, u_1^I, represents only a regular wave propagating along the x-axis and this potential must satisfy the first-order boundary-value problem when no rigid body is present. Therefore, ϕ_1^I satisfying Eqs. (17-18) and (20-21), is given by

$$u_1^I = -\frac{1}{a}\frac{\cosh\left[a(y+h)\right]}{\cosh(ah)} e^{iax} \tag{30}$$

The parameters a and ν are related through the well-known equation from linear wave theory.

$$\nu = \frac{\sigma^2\bar{a}}{g} = a \tanh(ah) \tag{31}$$

which, in dimensional form, using $a = k\bar{a}$, and $k = 2\pi/\bar{L}$ is:

$$\sigma^2 = gk\ \mathrm{tank}(\bar{k}h) \tag{32}$$

Since the solution for u_1^I is known, the boundary-value problem for u_1^S may now be established. Substituting Eqs. (30), (29) and (27) into the first-order problem given by Eqs. (17-21), and eliminating η_1 between Eqs. (20) and (21) results in a

boundary-value problem for the first-order scatter potential as follows:

$$\nabla^2 u_1^S(x,y) = 0 \tag{33}$$

$$u_{1y}^S(x,-h) = 0 \tag{34}$$

$$u_{1n}^S(x,y) = \frac{1}{\cosh(ah)}\left[n_y \sinh[a(y+h)] + i n_x \cosh[a(y+h)]\right] e^{iax}$$

$$\text{on } S_1(x,y) = 0 \tag{35}$$

$$u_{1y}^S(x,0) - \nu u_1^S(x,0) = 0 \tag{36}$$

where n_x and n_y denote the x- and y-components, respectively, of the unit normal vector, $\hat{n} = \hat{i}n_x + \hat{j}n_y$, directed outward into the fluid.

D. SECOND-ORDER BOUNDARY-VALUE PROBLEM

In view of the linearity of the second-order problem, the same procedure as used in the first-order problem may be applied and leads to the representation of the second-order potential as the sum:

$$\phi_2 = \phi_2^I + \phi_2^S \tag{37}$$

where ϕ_2^I denotes the second-order incident wave potential and ϕ_2^S denotes the scatter potential.

Considering the case where there is no body present, there is no scattered wave and therefore, $\phi_1^S = \phi_2^S = 0$. Additionally, the boundary conditions on the immersed cylinder, Eqs. (19) and (24), are not applicable. Thus, when the substitutions of ϕ_1^I for ϕ_1 and ϕ_2^I for ϕ_2 are made into Eqs. (22), (23), (25) and (26), along with Eqs. (20), (21), (27) and (31), and upon eliminating η_1 and η_2 between Eqs. (25) and (26), the resulting boundary-value problem for the second-order incident wave potential becomes:

$$\nabla^2 \phi_2^I(x,y,t) = 0 \tag{38}$$

$$\phi_{2y}^I(x,-h,t) = 0 \tag{39}$$

$$\phi_{2y}^I(x,0,t) + \nu\phi_{2tt}^I(x,0,t) = -\tfrac{3}{2}b^2(a^2-\nu^2)R_e[ie^{i2(ax-t)}] \tag{40}$$

The periodic solution to the incident-wave boundary-value problem specified by Eqs. (38-40) is known and may be expressed as

$$\phi_2^I = \tfrac{3}{4}ab^2\nu R_e[iu_2^I(x,y)e^{-i2t}] \tag{41}$$

where the second-order complex potential, u_2^I, is given by:

$$u_2^I(x,y) = -\frac{1}{2a}\frac{\cosh[2a(y+h)]}{\sinh^4(ah)}e^{i2ax} \tag{42}$$

Eqs. (41-42) are familiar in wave theory and are exactly the same form as that given by Ref. 4 for second-order Stokes' waves.

Having developed a solution for the second-order incident wave potential, ϕ_2^I, the problem for second-order scatter potential may be determined. To this end, $\phi_1 = \phi_1^I + \phi_1^S$ and $\phi_2 = \phi_2^I + \phi_2^S$ are substituted into Eqs. (22-26), and the previously determined solutions for the incident wave potentials, Eqs. (30) and (42), are utilized. After applying Eqs. (20), (21), (27), (29) and (41) and eliminating η_2 between Eqs. (25) and (26), the resulting boundary-value problem for the second-order scatter potential becomes:

$$\nabla^2 \phi_2^S(x,y,t) = 0 \tag{43}$$

$$\phi_{2y}^S(x,-h,t) = 0 \tag{44}$$

$$\phi_{2n}^S(x,y,t) = -\frac{3}{4}\frac{ab^2 R_e}{\sinh^4(ah)}\Big[(n_x \cosh[2a(y+h)]$$

$$- in_y \sinh[2a(y+h)])e^{i2(ax-t)}\Big] \tag{45}$$

$$\text{on } S_1(x,y) = 0$$

$$\phi_{2y}^S + \phi_{2tt}^S = \frac{a^2b^2}{2} R_e[i(\frac{1}{\nu}u_{1y}^S u_{1yy}^S + u_1^I u_{1yy}^S - 6u_{1y}^I u_{1y}^S \tag{46}$$

$$+ \frac{1}{\nu}u_{1y}^S u_{1yy}^I - 4u_{1x}^I u_{1x}^S - 2{u_{1x}^S}^2 - 3{u_{1y}^S}^2)e^{-i2t}]$$

$$+ U(x) \qquad \text{on } y = 0$$

where U(x) is a non-periodic function generated by the substitution of ϕ_1 into Eqs. (25) and (26). For brevity, this function is not written out since it is not needed in determining the second-order pressures and forces.

The boundary-value problem in ϕ_2^S given in Eqs. (43-46) is time dependent according to e^{-i2t}, except for the U(x) term in Eq. (46). Therefore, the solution for ϕ_2^S may be taken in the form

$$\phi_2^S = \frac{3}{4}\, ab^2 \nu R_e \left[i \left[u_2^S\,(x,y) e^{-i2t} + \tilde{u}_2^S(x,y) \right] \right] \tag{47}$$

where the last term denotes the time-independent portion of the complex potential. Separate boundary-value problems for u_2^S and $\tilde{u}_2^S$ arise from the substitution of Eq. (47) into Eqs. (43-46). However, as will be demonstrated only the ϕ_{2t}^S term is required for determination of the pressure to the second order. Therefore, the time-derivative of the time-independent part of ϕ_2^S will be zero, and, accordingly, will not be developed further here; the solution for u_2^S only will be pursued.

When Eq. (47) is substituted into Eqs. (43-46), the resulting boundary-value problem for the second-order scatter potential takes the form:

$$\nabla^2 u_2^S(x,y) = 0 \tag{48}$$

$$u_{2y}^S\,(x,-h) = 0 \tag{49}$$

$$u_{2n}^{S}(x,y) = \frac{1}{\sinh^4(ah)}\Big[n_y \sinh[2a(y+h)]$$

$$+ in_x \cosh[2a(y+h)]\Big] e^{i2ax} \quad \text{on } S_1(x,y)=0 \qquad (50)$$

$$u_{2y}^{S}(x,0) - 4\nu u_2^S(x,0) = f^*(x) \qquad (51)$$

where

$$f^*(x) = \frac{2a}{3\nu}[\frac{1}{\nu}u_{1y}^S u_{1yy}^S + u_1^I u_{1yy}^S - 6u_{1y}^I u_{1y}^S + \frac{1}{\nu}u_{1y}^S u_{1yy}^I \qquad (52)$$

$$- 4u_{1x}^I u_{1x}^S - 2{u_{1x}^S}^2 - 3{u_{1y}^S}^2]_{y=0}$$

There is considerable similarity in the first-order and second-order problems, the only difference in form being that the second-order free surface boundary condition, Eq. (51), is non-homogeneous.

Because of the non-homogenity of Eq. (51), further use is made of linear superposition defining u_2^S as the sum,

$$u_2^S = u_2^{S^o} + u_2^{S^*} \qquad (53)$$

where $u_2^{S^o}$ denotes the homogeneous solution and $u_2^{S^*}$ the particular solution to the boundary-value problem as stated in Eqs. (48-51). More precisely, $u_2^{S^*}$ and $u_2^{S^o}$ are defined as the solutions to the boundary-value problems obtained from the substitution of Eq. (53) into Eqs. (48-51) as follows:

Particular Solution (u_2^{S*}):

$$\nabla^2 u_2^{S*}(x,y) = 0 \tag{54}$$

$$u_{2y}^{S*}(x,-h) = 0 \tag{55}$$

$$u_{2y}^{S*}(x,0) - 4\nu u_2^{S*}(x,0) = f^*(x) \tag{56}$$

Homogeneous Solution ($u_2^{S^o}$)

$$\nabla^2 u_2^{S^o}(x,y) = 0 \tag{57}$$

$$u_{2y}^{S^o}(x,-h) = 0 \tag{58}$$

$$u_{2y}^{S^o}(x,0) - 4\nu u_2^{S^o}(x,0) = 0 \tag{59}$$

$$u_{2n}^{S^o}(x,y) = \frac{1}{\sinh^4(ah)} \Big[n_y \sinh[2a(y+h)] + i n_x \cosh[2a(y+h)] \Big] e^{i2ax} - u_{2n}^{S^*}(x,y) \tag{60}$$

on $S_1(x,y) = 0$

By the division of the second-order problem into homogeneous and particular solutions, the non-homogeneous problem for u_2^{S*} contains no boundary condition on the cylinder surface. The resulting boundary-value problem for u_2^{S*}, as stated in Eqs. (54-56), is identical to that associated with the linear problem for the potential

resulting from a harmonic pressure variation of amplitude distribution $f^*(x)$ on the free surface in water of depth h. The homogeneous boundary-value problem for $u_2^{s^o}$, defined by Eqs. (57-60), is similar in form to the first-order problem given by Eqs. (33-36), the significant difference being the term 4ν in place of ν which occurs in the free surface boundary condition. Thus, the method of solution for $u_2^{s^o}$ will be similar to that of the first-order scatter potential

E. DEFINITION OF THE INCIDENT WAVE

Having developed the first-order and second-order boundary-value problems, it is now appropriate to completely define the incident wave height and in the process determine expressions for the unknown constants b and K_2. Solving Eqs. (21) and (26) for η_1 and η_2, respectively, and then substituting the results into the expression for the free surface elevation as given by Eq. (14) yields:

$$\eta(x,t) = -\epsilon\phi_{1t} + \epsilon^2[-\phi_{2t} + \phi_{1t}\phi_{1yt} + \phi_{1tt}\phi_{1y} - \frac{1}{2}(\phi_{1x}^2 + \phi_{1y}^2) + K_2] + O(\epsilon^3) \tag{61}$$

where the potentials are evaluated at y = 0. Eq. (61) expresses the free surface elevation in terms of the total potential, and therefore, includes the effects of both the incident and the scattered waves. Hence, as it is of interest to obtain an expression for the free surface elevation

of the incident wave, the scatter potentials are set to zero, i.e. $\phi_1^S = \phi_2^S = 0$. Evaluating Eq. (61) using the known solutions for the incident wave potentials, Eqs. (30) and (42), along with Eqs. (27) and (41), then yields:

$$\eta^I(x,t) = \varepsilon b R_e[e^{i(ax-t)}] + \varepsilon^2 \Bigg[\frac{b^2(\nu^2-a^2)}{4\nu} + K_2 + \frac{ab^2}{4}\,\frac{\cosh(ah)\,[2+\cosh(2ah)]}{\sinh^3(ah)}\; R_e[e^{i2(ax-t)}]\Bigg] + 0(\varepsilon^3) \tag{62}$$

where $\eta^I(x,t)$ denotes the dimensionless surface elevation for the incident wave with no cylinder present. If the x-axis is placed at the mean water line then the constant term must vanish and, therefore,

$$K_2 = \frac{b^2(a^2-\nu^2)}{4\nu} \tag{63}$$

The remaining terms in Eq. (62) are time-dependent, periodic functions, the second-order term having a frequency of twice that of the first-order.

Defining the dimensionless wave height given in Eq. (7) as the difference in elevation between the crest and trough of the incident wave, εb must represent the wave amplitude. The second-order term in Eq. (62) having frequency twice the fundamental frequency makes equal contributions to surface

elevation at both the crest and trough, so that it contributes nothing to the wave height. Thus, in terms of dimensionless wave height, b is given by

$$H = \varepsilon b = \overline{H}/2\overline{a} \qquad (64)$$

where $\overline{H}$ denotes the elevation difference between the wave crest and trough.

Expressing the incident wave profile in terms of the dimensionless wave height yields:

$$\eta^{I}(x,t) = H \cos (ax-t) \qquad (65)$$

$$+ \frac{H^2 a \cosh(ah)}{4 \sinh^3(ah)} [2 + \cosh(ah)] \cos[2(ax-t)]$$

It may be noted that Eq. (65) agrees with the expression for the second-order Stokes' wave given, for example, by Ref. 4.

F. PRESSURES AND FORCES

The pressure may be formulated by use of Bernoulli's equation arranged as follows:

$$P(\overline{x},\overline{y},\overline{t}) = -\rho\Phi_{\overline{t}} - \frac{1}{2}\rho[\Phi_{\overline{x}}^{2} + \Phi_{\overline{y}}^{2}] - \rho g\overline{y} + \rho g\overline{K} \qquad (66)$$

where $P(\overline{x},\overline{y},\overline{t})$ denotes the dimensional pressure and ρ denotes the fluid density. By use of Eqs. (7), (13-16), (27), (41),

(47), and (64), Eq. (66) may be reduced to dimensionless form, and carried to the second-order in wave height. The result is

$$p(x,y,t) = -y - HaR_e[u_1 e^{-it}] \tag{67}$$

$$- \frac{H^2 a^2}{4\nu}\Big[R_e[(\frac{6\nu^2}{a}u_2 - u_{1x}^2 - u_{1y}^2)e^{-i2t}]$$

$$+ |u_{1x}|^2 + |u_{1y}|^2 + \frac{\nu^2}{a^2} - 1\Big]$$

In Eq. (67) $p(x,y,t)$ denotes the dimensionless pressure coefficient and is defined as:

$$p(x,y,t) = \frac{P(\bar{x},\bar{y},\bar{t})}{\rho g \bar{a}} \tag{68}$$

The first term in Eq. (67) represents the hydrostatic pressure as y is the dimensionless depth beneath the mean free surface ($y = 0$). The second and third terms are harmonic, the third term having twice the frequency of the second, and representing the harmonic second-order contribution. The remaining terms in Eq. (67) are independent of time and result in the time-average or steady-state force components.

Expressing the components of the wave force vectors in terms of integrals of the pressure over the cylinder surface area, the dimensionless force coefficients may be written as

$$C_i(t) = -\int_{S_1} p(x,y,t)\, n_i \, dS_1, \quad i = 1,2 \tag{69}$$

where the dimensionless force coefficients in the x and y directions are defined, respectively, as

$$C_1(t) = \frac{F_x(t)}{\rho g \bar{a}^3} \tag{70}$$

$$C_2(t) = \frac{F_y(t)}{\rho g \bar{a}^3} \tag{71}$$

where $F_x(t)$ and $F_y(t)$ denote the force components. Additionally, dS_1 denotes a dimensionless differential arc length along the circumference of the cylinder.

Applying Eq. (67) to Eq. (69) yields:

$$C_i(t) = - y \int_{S_1} n_i dS_1 - Ha \int_{S_1} R_e[u_1 e^{-it}]n_i \, dS_1 \tag{72}$$

$$- H^2 [\frac{a^2}{4\nu} \int_{S_1} R_e[\frac{6\nu^2}{a} u_2 - u_{1x}^2 - u_{1y}^2) e^{-i2t}$$

$$+ |u_{1x}|^2 + |u_{1y}|^2 + \frac{\nu^2}{a^2} - 1]n_i dS_1 + 0(H^3)$$

The first term in Eq(72) results from the hydrostatic pressure on the cylinder so it represents simply the buoyancy force. Disregarding the hydrostatic pressure the hydrodynamic force may be written as

$$C_i(t) = H F_{1i} \cos(\delta_{1i} - t) + H^2 \tag{73}$$

$$+ H^2 [F_{2i} \cos(\delta 2i - 2t) + F_{2i}^{SS}] + O(H^3)$$

where the first-order, second-order, and steady-state force coefficients and phase shift angles are defined by comparison of Eqs. (72) and (73) as follows:

$$F_{1i} e^{i\delta_{1i}} = a \int_{S_1} u_1 n_i dS_1 \tag{74}$$

$$F_{2i} e^{i\delta_{2i}} = \frac{a^2}{4\nu} \int_{S_1} \left(\frac{6\nu^2}{a} u_2 - u_{1x}^2 - u_{1y}^2\right) n_i dS_1 \tag{75}$$

$$F_{2i}^{SS} = \frac{a^2}{4\nu} \int_{S_1} \left(|u_{1x}|^2 + |u_{1y}|^2 + \frac{\nu^2}{a^2} - 1\right) n_i dS_1 \tag{76}$$

The dimensionless force coefficients, F_{1i} and F_{2i} are real.

The first term in Eq. (73) represents the first-order forces of the fundamental frequency, σ. The periodic portion of the second term in Eq. (73) represents the second-order contribution to the force at twice the fundamental frequency. The last second-order term represents the steady-state or time-independent contribution, sometimes called drift-force.

Evaluation of the force coefficients defined by Eqs. (74-76) is the main objective of this work. Therefore, it is necessary to evaluate the first-order and second-order complex potentials, u_1 and u_2, as well as derivatives of u_1

on the surface of the cylinder. Additionally, evaluation of u_1 and its derivatives on the mean free surface $(y = 0)$ is required in order to carry out the solutions.

III. METHOD OF SOLUTION AND NUMERICAL PROCEDURES

A. SOLUTION OF THE FIRST-ORDER PROBLEM

The use of a Green's function to express the solution to the first-order boundary-value problem was first formulated by John [Ref. 5], and applied to submerged ellipsoids by Garrison and Rao [Ref. 3]. This method is considered to be applicable, in principle, to arbitrary shapes and is mathematically the most straightforward. The Green's function, G, of unit strength which satisfies Eq. (33) as well as the boundary conditions, Eqs. (34) and (36) is given by:

$$G(x,y;\xi,\eta;\nu) = \ln r - \ln r' + 2PV \int_0^\infty \left[\frac{\cosh[\mu(y+h)]\cosh[\mu(\eta+h)]}{(\cosh\mu h)(\nu\cosh\mu h-\mu\sinh \mu h)}\right.$$

$$\left. - e^{-\mu h} \frac{\sinh(\mu\eta)\sinh(\mu y)}{\sinh(\mu h)}\right] \cos|x-\xi| d\mu \tag{77}$$

$$- i \frac{4\pi}{2a_1 h+\sinh(2a_1 h)} \cosh[a_1(y+h)]\cosh[a_1(\eta+h)]\cos[a_1|x-\xi|]$$

where:

$$r = [(x-\xi)^2 + (y-\eta)^2]^{\frac{1}{2}} \tag{78}$$

$$r' = [(x-\xi)^2 + (y+\eta)^2]^{\frac{1}{2}} \tag{79}$$

The symbol, a_1, is defined in terms of h and ν as the solution to the equation

$$F(a_1,h,\nu) = 0 \tag{80}$$

where F is defined by

$$F(a_1,h,\nu) = a_1 \tanh(a_1 h) - \nu \tag{81}$$

Comparison of Eqs. (31), (80) and (81) demonstrates that a_1 is clearly the equivalent of a. However, this will not be the case for the second-order problem, thus, requiring the use of separate notation. In Eq. (77), PV denotes the principal value of the integral.

The Green's function given in Eq. (77) was also given in series form by John [Ref.5] as

$$G(x,y;\xi,\eta;\nu) = 2\pi \sum_{k=1}^{\infty} \frac{(\mu_k^2+\nu^2)\cos[\mu_k(y+h)]}{\nu\mu_k - h\mu_k(\mu_k^2+\nu^2)} \cos[\mu_k(\eta+h)]e^{-\mu_k|x-\xi|}$$

$$- i\,\frac{4\pi\cosh[a_1(y+h)]\cosh[a_1(\eta+h)]e^{ia_1|x-\xi|}}{2a_1 h + \sinh(2a_1 h)} \tag{82}$$

where a_1 is as defined in Eq. (80) and the quantities μ_k are defined as the real positive roots of the equation

$$K(\mu_k,h,\nu) = 0 \tag{83}$$

where the function K is defined as

$$K(\mu,h,\nu) = \mu \tan(\mu h) + \nu \tag{84}$$

Following the Green's function method of solution, u_1^S is written as the integral over the cylinder arc length, S_1, as:

$$u_1^S(x,y) = \frac{1}{2\pi} \int_{S_1} f_1(\xi,\eta) G(x,y;\xi,\eta;\nu)\, dS_1 \tag{85}$$

where (ξ,η) denotes points on the immersed surface, $f_1(\xi,\eta)$ denotes the unknown source strength, and $dS_1 = d\bar{S}_1/\bar{a}$ denotes the differential arc length on the cylinder surface, made dimensionless with the characteristic length, $\bar{a}$.

Although the solution to the first-order boundary-value problem as stated in Eqs. (33-36) is given by Eq. (85), the source strength function, $f_1(\xi,\eta)$, must be determined in order to evaluate the potential. From potential theory, the normal derivative of the potential, $u_1^S(x,y)$, on the surface of the cylinder is

$$u_{1n}^S(x,y) = \frac{1}{2} f_1(x,y) + \frac{1}{2\pi} \int_{S_1} f_1(\xi,\eta) G_n(x,y;\xi,\eta;\nu)\, dS_1 \tag{86}$$

where G_n (the normal derivative of G) may be determined by differentiation of either Eq. (77) or (82) in a straightforward manner.

Applying the final boundary condition, Eq. (35), leads to an integral equation which may be solved for f_1,

$$\frac{1}{2} f_1(x,y) + \frac{1}{2\pi} \int_{S_1} f_1(\xi,\eta) G_n(x,y;\xi,\eta,\nu)\, dS_1 \tag{87}$$

$$. = \frac{1}{\cosh(ah)} \left[n_y \sinh[a(y+h)] + i n_x \cosh[a(y+h)] \right] e^{iax}$$

The solution for f_1 from Eq. (87) may then be used in Eq. (85) to determine the potential, u_1^S.

B. SOLUTION OF THE SECOND-ORDER PROBLEM

The similarity in form of the boundary-value problem for the homogeneous part of the second-order potential, Eqs. (57-60), to the first-order potential is now utilized. Since the only significant difference occurs in Eq. (59), where ν is replaced by 4ν, we may represent $u_2^{S^O}$ in a form similar to Eq. (85) as

$$u_2^{S^O}(x,y) = \frac{1}{2\pi} \int_{S_1} f_2(\xi,\eta) G(x,y;\xi,\nu;4\nu)\, dS_1 \tag{88}$$

where $G(x,y;\xi,\eta;4\nu)$ is defined by replacing ν by 4ν in Eqs. (77) and (82). Therefore, a_2 is defined by

$$F(a_2,h,4\nu) = 0 \tag{89}$$

and a_1 is replaced by a_2 in Eqs. (77) and (82), also. In the case of Eq. (82), μ_k is defined as the real positive roots of

$$K(\mu_k, h, 4\nu) = 0 \tag{90}$$

The source strength function, $f_2(\xi,\eta)$, appearing in Eq. (88) is again determined by applying the kinematic boundary condition on the cylinder surface as given by Eq. (59). The integral equation for f_2 may then be given by

$$\frac{1}{2} f_2(x,y) + \frac{1}{2\pi} \int_{S_1} f_2(\xi,\eta) G_n(x,y;\xi,\eta;4\nu)\, dS_1 \tag{91}$$

$$= \left[\frac{n_y \sinh[2a(y+h)] + in_x \cosh[2a(y+h)]}{\sinh^4(ah)}\right] e^{i2ax} - u_{2n}^{S*}(x,y)$$

$$\text{on } S_1(x,y) = 0$$

However, to solve Eq. (91) for f_2, first a solution for u_2^{S*} must be obtained and u_{2n}^{S*} evaluated on the cylinder surface. Thus, at this point the solution to u_2^{S*} is sought.

The complex potential u_2^{S*} is defined as the solution to the boundary-value problem, Eqs. (54-56). The form of this problem is recognized as being equivalent to that of fluid motion produced by a periodic pressure distribution (as specified by $f^*(\xi)$) with frequency 2σ on the free surface. No cylinder is considered to exist in the fluid region.

The solution to this problem may be formulated as an integral over the free surface extending from negative infinity to positive infinity. Using the present

dimensionless representation, u_2^{S*} becomes:

$$u_2^{S*}(x,y) = \frac{1}{2\pi} \int_{S_2} f^*(\xi) G^*(x,y;\xi,0;4\nu)\, d\xi \tag{92}$$

The Green's function, G^*, for this problem is given in Ref. 8. Upon transforming the solution to the non-dimensional form

$$G^*(x,y;\xi,0;4\nu) = -2PV \int_{-\infty}^{\infty} \frac{\cosh[\mu(y+h)]\cos[\mu(x-\xi)]d\mu}{4\nu\cosh(\mu h) - \mu\sinh(\mu h)}$$

$$-i\, \frac{4\pi\cosh[a_2h]\cosh[a_2(y+h)]\cos[a_2(x-\xi)]}{2a_2h + \sinh(2a_2h)} \tag{93}$$

where a_2 is defined by

$$F(a_2,h,4\nu) = 0 \tag{94}$$

The normal derivative of u_2^{S*} is required in order to completely define the kinematic boundary condition on the cylinder as given in Eq. (60). From Eq. (92), this becomes

$$u_{2n}^{S*}(x,y) = \frac{1}{2\pi} \int_{S_2} f^*(\xi) G_n^*(x,y;\xi,0;4\nu)\, d\xi \tag{95}$$

The derivative of G^* in the direction normal to the cylinder surface may be obtained by differentiation of Eq. (93) in a straightforward manner. Thus, using the values for $f^*(\xi)$, as defined in terms of the first-order solution only, and as

given in Eqs. (52) and (56), u_2^{S*} and u_{2n}^{S*} may be determined. Using u_{2n}^{S*}, $u_2^{S^o}$ may then be determined to complete the second-order boundary-value problem solution.

C. NUMERICAL METHODS

Determination of the forces on the cylinder surface requires solving for the potentials u_1, u_2 and the derivatives of u_1 at points on the surface, as shown in Eq. (72). Thus, the problem is now one of solving for both the first-order and the second-order scatter potentials, and requires solving for the source strength functions f_1 and f_2, as well as the free surface pressure distribution source strength function, f^*. In view of the complexity of the equations it is natural to attempt a numerical solution.

The first-order scattering potential, u_1^S, as given by Eq. (85) is dependent upon the first-order source strength function, f_1. Thus, it is necessary to solve Eq. (87) for f_1 to determine u_1^S. A numerical solution may be developed by dividing the cylinder surface into elements of length $\Delta\theta = 2\pi/m$, with the center of each element assigned an index. Since $f_1(\xi,\eta)$ is recognized as a well-behaved function for a smooth surface cylinder, it is appropriate to define parameters:

$$\alpha_{ij}(\nu) = \frac{1}{\pi} \int_{\Delta S_{1j}} G_n(x_i, y_i; \xi, \eta; \nu)\, dS_1 \qquad (96)$$

$$i,j = 1,2,3,\ldots,m$$

$$h_i = \frac{1}{\cosh(ah)} \Big[n_y(x_i,y_i) \sinh[a(y_i+h)] \qquad (97)$$

$$+ i n_x(x_i,y_i) \cosh[a(y_i+h)] \Big] e^{iax_i}$$

$$f_{1i} = f_1(x_i,y_i) \qquad (98)$$

and, thus, Eq. (87) may be approximated by the complex matrix equation,

$$f_{1i} + \alpha_{ij}(\nu) f_{1j} = 2h_{1j} \qquad i,j = 1,2,\ldots,m \qquad (99)$$

Upon evaluation of $\alpha_{ij}(\nu)$ using Eq. (96), the inversion of Eq. (99) may be carried out on the digital computer to determine f_{1j} at each nodal point on the surface of the cylinder.

For the purpose of evaluating α and β, either Eq. (77) or Eq. (82) may be used. For evaluation of α and β when $i = j$ Eq. (77) must be used to allow separate treatment of the logarithmic singularity as r approaches 0. The difficulty encountered with the logarithmic singularity may be overcome by carrying out its integration analytically. Garrison [Ref. 2] showed for the calculation of α_{ij} that the contribution of the normal derivative of the ln (r) in Eq. (77) integrated over the singular element of $\Delta\theta$ is $\Delta\theta/2$, not only for the nodal point $(x_i,y_i) = (\xi,\eta)$, but for all nodal points

(x_i, y_i) on the cylinder surface. Additionally, to calculate β_{ij} Garrison has developed a numerical approximation for the integral of the ln (r) over the singular element of $\Delta\theta$.

A second singularity arises in the evaluation of the infinite integral occurring in Eq. (77). The first term in the integrand is singular at $\mu = a$. Since the numerator approaches zero like $(\mu-a)$ near a, Garrison [Ref. 2] subtracted $1/(\mu-a)$ from the singular term in the range $0 \leq \mu \leq 2a$ and added the contribution of the integral of $1/(\mu-a)$ (which in this case was zero). This converted the singular integrand to a regular function. Applying this technique to numerically evaluate the resulting integral between 0 and 2a, Simpson's three-eights rule is used with an odd number of subdivisions, thus ensuring that the point $\mu = a$ is not encountered.

With f_1 determined from the inversion of Eq. (99), the first-order potential and its derivatives may be evaluated on the cylinder surface. Replacing the surface integrals with summations, Eq. (85) and its derivatives may be written as

$$u^S_{1i} = \beta_{ij}(\nu) f_{1j} \qquad i,j = 1,2,3,\ldots,m \tag{100}$$

$$u^S_{1xi} = \beta_{xij}(\nu) f_{1j} \tag{101}$$

$$u^S_{1yi} = \beta_{yij}(\nu) f_{1j} \tag{102}$$

where u^S_{1i}, u^S_{1yi}, etc. denote functions evaluated at the i^{th} nodal point on the cylindrical surface. The complex matrices in Eqs. (100-102) are defined as follows:

$$\beta_{ij}(\nu) = \frac{1}{2\pi} \int_{\Delta S_{1j}} G(x_i,y_i;\xi,\eta;\nu)\,dS_1 \qquad i,j = 1,2,\ldots,m \quad (103)$$

$$\beta_{xij}(\nu) = \frac{1}{2\pi} \int_{\Delta S_{1j}} G_x(x_i,y_i;\xi,\eta;\nu)\,dS_1 \quad (104)$$

$$\beta_{yij}(\nu) = \frac{1}{2\pi} \int_{\Delta S_{1j}} G_y(x_i,y_i;\xi,\eta;\nu)\,dS_1 \quad (105)$$

The first-order incident potential and its derivatives may be evaluated directly at each nodal point after differentiation of Eq. (30) as needed, and combined with the scattering potential and its respective derivatives.

To solve the second-order problem, the function $f^*(\xi)$ required in Eq. (95), and defined by Eq. (52), must be evaluated numerically. Dividing the mean free surface ($y = 0$) in the vicinity of the cylinder into n equal increments, f^* is evaluated at each nodal point. The numerical solution again uses Eqs. (100-105) with $y_i = 0$ for the purpose of evaluating u^S_{1i} and its derivatives at the mean free surface nodal points. The first-order wave potential and its derivatives are evaluated directly at each nodal point, and combined with the scatter potential and its derivatives to solve Eq. (52) for f^*.

Having determined f* at all nodal points on the free surface, the boundary-value problem for u_2^{S*} given by Eqs. (54-56) may be solved, i.e. u_2^{S*} and u_{2n}^{S*} may be evaluated on the cylinder by solution of Eqs. (92) and (95) respectively. Expressing the integrals as numerical summations in complex matrix form yields:

$$u_{2ni}^{S*} = f_j^* \alpha_{ij}^*(4\nu) \qquad \begin{array}{l} i = 1,2,\ldots,m \\ j = 1,2,\ldots,n \end{array} \tag{106}$$

$$u_{2i}^{S*} = f_j^* \beta_{ij}^*(4\nu) \tag{107}$$

where

$$\alpha_{ij}^*(4\nu) = \frac{1}{\pi} \int_{\Delta S_{2j}} G^*(x_i, y_i; \xi, \eta; 4\nu)\ d\xi \tag{108}$$

$$\beta_{ij}^*(4\nu) = \frac{1}{2\pi} \int_{\Delta S_{2j}} G_n^*(x_i, y_i; \xi, \eta; 4\nu)\ d\xi \tag{109}$$

Thus, using f*, both u_2^{S*} and u_{2n}^{S*} may be directly evaluated at each cylinder surface nodal point.

To solve the boundary-value problem for the second part of the second-order scatter potential, $u_2^{S^o}$, as specified by Eqs. (57-60), Eq. (88) must be evaluated. However, to evaluate Eq. (88) for $u_2^{S^o}$, the second-order source strength function, f_2, must be evaluated by numerically solving

the integral equation, Eq. (91). Replacing the integral equation with a complex matrix summation:

$$f_{2i} + f_{2j}\alpha_{ij}(4\nu) = 2k_i \qquad i,j = 1,2,\ldots,m \tag{110}$$

where $\alpha_{ij}(4\nu)$ is defined by Eq. (96),

$$f_{2j} = f_2(x_j,y_j) \tag{111}$$

and,

$$k_i = \frac{1}{\sinh^4(ah)}\Big[n_y(x_i,y_i)\sinh[2a(y_i+h)] \tag{112}$$

$$+ in_x(x_i,y_i)\cosh[2a(y_i+h)]\Big]e^{i2ax_i} - u_{2n}^{S*}(x_i,y_i)$$

Once $\alpha_{ij}(4\nu)$ is evaluated, Eq. (110) may be solved to obtain the second-order source strength function, f_2, at each nodal point on the cylinder surface. Stating Eq. (88) in summation form,

$$u_{2i}^{S^o} = f_{2j}\beta_{ij}(4\nu) \qquad i,j = 1,2,3,\ldots,m \tag{113}$$

where $\beta_{ij}(4\nu)$ is defined by Eq. (103). Using Eq. (113), $u_2^{S^o}$ may be determined at each nodal point on the cylinder surface, then combined with u_2^{S*} and the second-order incident potential, u_2^I, to evaluate the total second-order potential, u_2.

Since the first-order potential, u_1, and its derivatives, u_{1x} and u_{1y}, as well as the second-order potential, u_2, are now known at each nodal point on the surface, the first-order, second-order and steady-state force coefficients and phase shift angles may be determined using Eqs. (74-76). As shown earlier, the integrals may be replaced by summations, using nodal point values and the arc length increment, $\Delta\theta$. Once each integral is evaluated, then the force coefficient becomes the absolute value or modulus of the complex integral result and the phase shift angle is the angle whose tangent is the imaginary part over the real part of the complex integral result.

D. COMPUTER SOLUTION

In order to utilize the method of solution described in Sections B and C, a computer program was developed to carry out the indicated calculations. Although accuracy was a prime consideration, an equally important requirement was to minimize computer run time. To achieve this, every attempt was made to utilize symmetry and to generate certain constants and matrices to eliminate redundant computations.

The program consists of a main program that flows in the order of computation presented in Sections B and C, along with four subroutines. Two subroutines to solve the first-order Green's function, using both forms of the function, Eqs. (77) and (82), were developed by Garrison [Ref. 2]. These subroutines, GREEN and GREENS respectively, have been

modified to include calculation of the first-order scatter potential derivatives, evaluation of the first-order scatter potential and its derivatives on the mean free surface, $y = 0$, and evaluation of the modified Green's function, G^*, for the determination of u_2^{S*} and u_{2n}^{S*} at each cylinder nodal point.

For elements of the α and β matrices corresponding to small values of the parameter $(x-\xi)$, GREEN is used while for larger values of $(x-\xi)$, GREENS is used. With the exception of the diagonal elements in Eqs. (96) and (103-105) and a few surface nodal points in Eqs. (108-109), the majority of elements of α and β are calculated by GREENS. This is most fortunate as the series form converges rapidly, requiring much less computer time than the equivalent integral form.

Subroutine GEODAT reads the input geometrical data, generates the matrices h_i and k_i as defined by Eqs. (97) and (112) respectively, and calculates certain geometrical parameters and matrices for repeated use in GREEN and GREENS. Subroutine COMAT inverts complex matrix equations and, therefore is used to invert Eqs. (99) and (110) to determine f_1 and f_2 respectively.

A cross-reference between text and computer program nomenclature is given in Table I.

TABLE I: Computer Program — Text Symbol Cross-Reference

Text	Computer Program	Text	Computer Program
a	A	n_x	ANX(I)
d	D	n_y	ANY(I)
f_1	F(I,1)	u_1	U1(I)
f_2	F1(I,1)	u_{1x}	U1X(I)
f*	FS(L)	u_{1y}	U1Y(I)
F_{11}	C1(1)	u_1^I (surface)	U1IS
F_{12}	C1(2)	u_{1x}^I (surface)	U1ISX
F_{21}	C2(1)	u_{1y}^I (surface)	U1ISY
F_{22}	C2(2)	u_{1yy}^I (surface)	U1ISYY
F_{21}^{SS}	C3(1)	u_1^S (surface)	U1SS
F_{22}^{SS}	C3(2)	u_{1x}^S (surface)	U1SSX
G	GIJ,GIJEXT	u_{1y}^S (surface)	U1SSY
G*	GIJ,GIJEXT	u_{1yy}^S (surface)	U1SSYY
G_n	GNIJ,GNJI	u_2	U2(I)
G_x	GXIJ,GXJI	x	X(I)
G_y	GYIJ,GYJI	y	Y(I)
G_{yy}	GYY	α	ALPHA(I,J)
h	H	β	BETA(I,J)
$2h_i$	HH(I,1)	β_x	BETAX(I,J)
k_i	PK(I,1)	β_y	BETAY(I,J)
m	NPTS	$\Delta\theta$ (cylinder)	DELTHE
n	NSPTS	$\Delta\xi$ (surface)	DELX

Text	Computer Program	Text	Computer Program
δ_{11}	PHASE1(1)	μ	AMU
δ_{12}	PHASE1(2)	$\mu_k(\nu)$	AMU(K)
δ_{21}	PHASE2(1)	$\mu_k(4\nu)$	AMU4(K)
δ_{22}	PHASE2(2)	ν	ANU
ξ	X(J)	4ν	ANU4
η	Y(J)	π	PI

IV. DISCUSSION AND RESULTS

A. SELECTION OF COMPUTER PROGRAM PARAMETERS

The computer program was developed on the IBM-360 computer using FORTRAN H. All subroutines were compiled on DATA CELL to minimize compile time and core size. In order to maximize accuracy while minimizing computational time, two key computer parameters were evaluated to determine optimum values: cylinder nodal points and free surface model points.

Using the first-order solution only, the number of nodal points on the cylinder surface was varied from 12 to 60 and compared with the results determined by Garrison [Ref. 2]. The minimum number of cylinder surface nodal points that provided accuracy within one percent of those obtained by Garrison using 60 points, was 24. Accordingly, the good correlation from 60 points down to 24 points led to the selection of 24 nodal points on the cylinder surface.

Since the free surface integral was to be carried out from $-\infty$ to $+\infty$, the next step was to establish the finite interval of convergence on the surface as well as the subdivision size. After the outer limits of the free surface integral were determined, the subdivision size was varied from 4 to 128 subdivisions per second-order wave length holding the total interval constant. Above 64 subdivisions

the results did not vary more than one percent, leading to the selection of 64 subdivisions per second-order wave length on the free surface for purposes of making calculations.

Additionally, the total interval, 2λ, was varied from one to eight wave lengths in both the positive and negative x directions. The second-order force coefficients and their respective phase shift angles converged to a small value varying periodically. Convergence to this limit cycle occurred between the first and second wave length from the center of the cylinder. Therefore, a total interval of from -2 to +2 second-order wave lengths was chosen as adequate for generation of the numerical results.

Although the values of the second-order force coefficients and their respective phase shift angles tended to converge with increasing the limits of the free surface integration, there remained, in general, the small limit cycle as noted above. The amplitude of the cycle was a function of the parameter, a, and was of significant magnitude only for values of a less than 0.25. The effects of a on the limit cycle for one of the second-order parameters, horizontal force coefficient, is demonstrated in Fig. (2). This represents a plot of percent variation of the force coefficient from the mean versus the surface interval size, $\lambda/(L/2)$, corresponding to values of a of 0.25, 0.20 and 0.15. Figure (2) is representative of the behavior of all second-order force coefficients and phase shift angles and demonstrates the

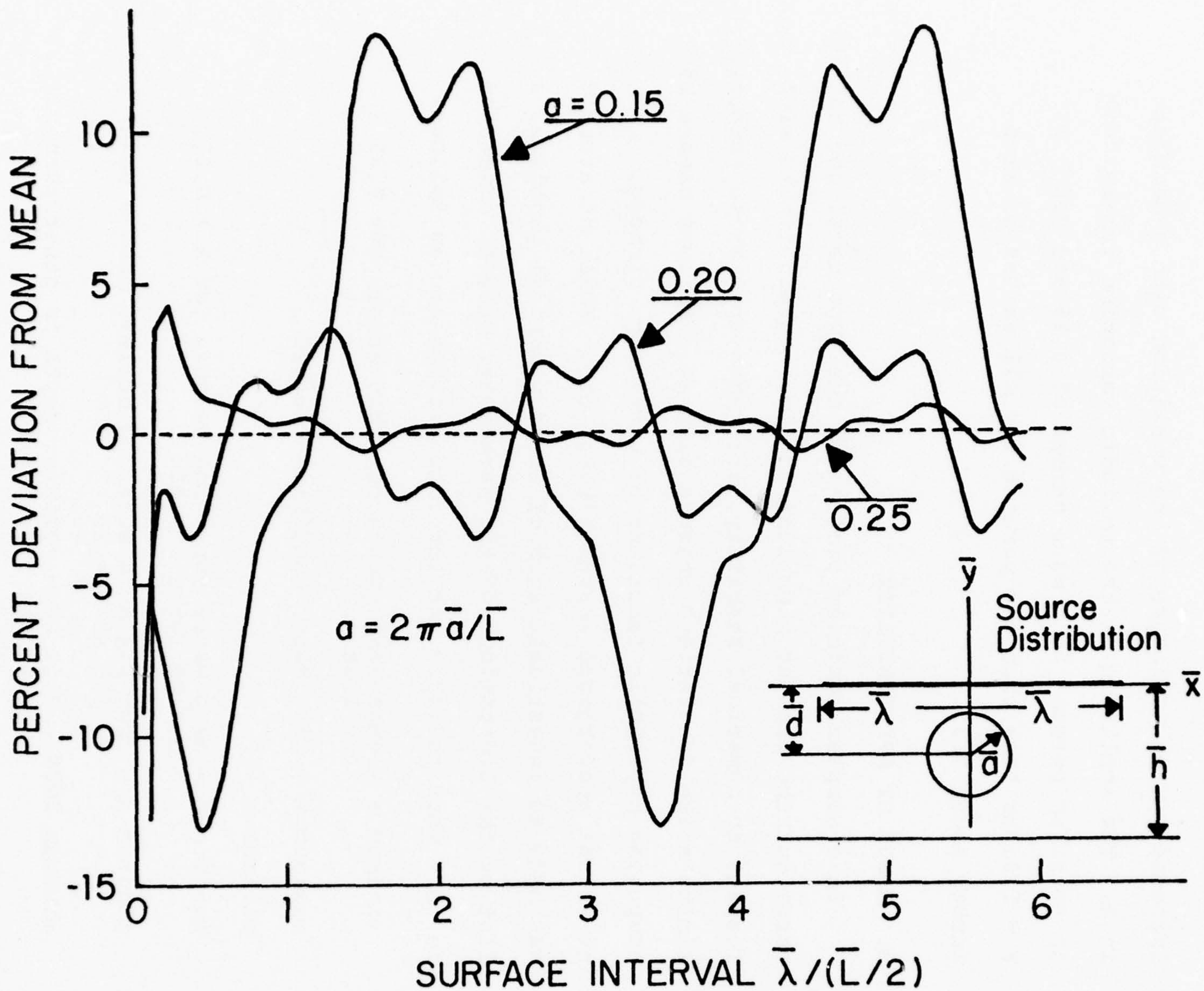

Figure 2. SECOND-ORDER HORIZONTAL WAVE FORCE COEFFICIENT VERSUS SURFACE INTERVAL
d = 2.0, h = 5.0

increase in the amplitude of the variation with a decrease in a. The amplitude of the periodic variation ranges from up to fifty percent at a wave number of 0.15 and depth of d = 1.4, to less than two percent for all values of wave number above 0.25.

B. *RANGE OF APPLICABILITY*

In a complex computer program of the type developed to carry out the present numerical scheme, certain limits with respect to numerical stability, etc. show up. These computing limits arise for various reasons such as overflows caused by computing hyperbolic functions of very large numbers, numerical convergence instabilities, etc. While no attempt was made to investigate each of these numerical problems, a list of the limitations of the particular program developed in this work to carry out the computations are as follows:

minimum a – dependent upon acceptable error, but 0.25 or less

maximum a – deep water condition reached

minimum h – 3

maximum h – deep water condition reached for $a > 0.25$, $h = 20$ for $a < 0.25$

minimum d – from 1.4 at h equal to or less than 5 down to 2.5 at h = 20

minimum SMIN – 0.12 when cylinder depth is other than near the free surface

maximum SMIN – 0.30 when cylinder depth is near the free surface

C. RESULTS

A representative set of results for the first-order and second-order force coefficients have been generated for a water depth of h = 5.0, submergence depths of d = 1.5, 2.0, 2.5, and 3.0, with a ranging from 0.15 to 1.2. All of the numerical results presented herein were based on 24 subdivisions on the circular cylinder and 64 subdivisions per second-order wave length on the free surface integration. The surface integral was carried out from x = -L to +L since this was found to be adequate for convergence. Since the second-order wave length is half the first-order wave length, L, the total interval, is equal to four second-order wave lengths.

The first-order horizontal and vertical force coefficients are presented in Figs. (3) and (4), respectively. It may be noted that, in general, the forces decrease with depth of submergence according to expectation since the wave action dies out with depth.

The second-order horizontal and vertical force coefficients are presented in Figs. (5) and (6), respectively. Generally, the results show the second-order effect to be relatively more important as the cylinder approaches the free surface. It might be suspected from this that the second-order contribution would be much more significant in the case of surface piercing or floating bodies.

The horizontal and vertical steady-state force coefficients are shown in Figs. (7) and (8), respectively. These results show that the horizontal steady-state force coefficient is very small in general. The horizontal force can be shown to be proportional to the momentum flux of the reflected wave and the reflected wave is small. In fact, in the case of infinite water depth, Dean [Ref. 1] showed that the reflection was exactly zero, and, accordingly, the steady-state horizontal force is likewise zero.

It may be noted that the steady-state vertical force is positive. This results from the fact that the velocities are largest on the top of the cylinder and, accordingly, the average pressure is reduced on the top of the cylinder.

The phase shift angles of the first-order and second-order forces are shown in Figs. (9) - (12).

To demonstrate the effect that the second-order terms have on the forces on the cylinder and their respective phase shift angles, a comparison was made between the first- and second-order results. Figures (13) - (18) are plots of the horizontal and vertical dimensionless forces on the cylinder versus time over one complete wave cycle for both the first-order solution alone and the combined first-order and second-order effects. Additionally, a plot of the incident wave is included to demonstrate the phase shift involved (the amplitude of the incident wave has no significance). A mean water depth, h, of 5.0, a cylinder depth, d, of 1.5 and a wave height, H, of 0.5 were used, with the wave number, a, assigned three values, 0.25, 0.5 and 1.0.

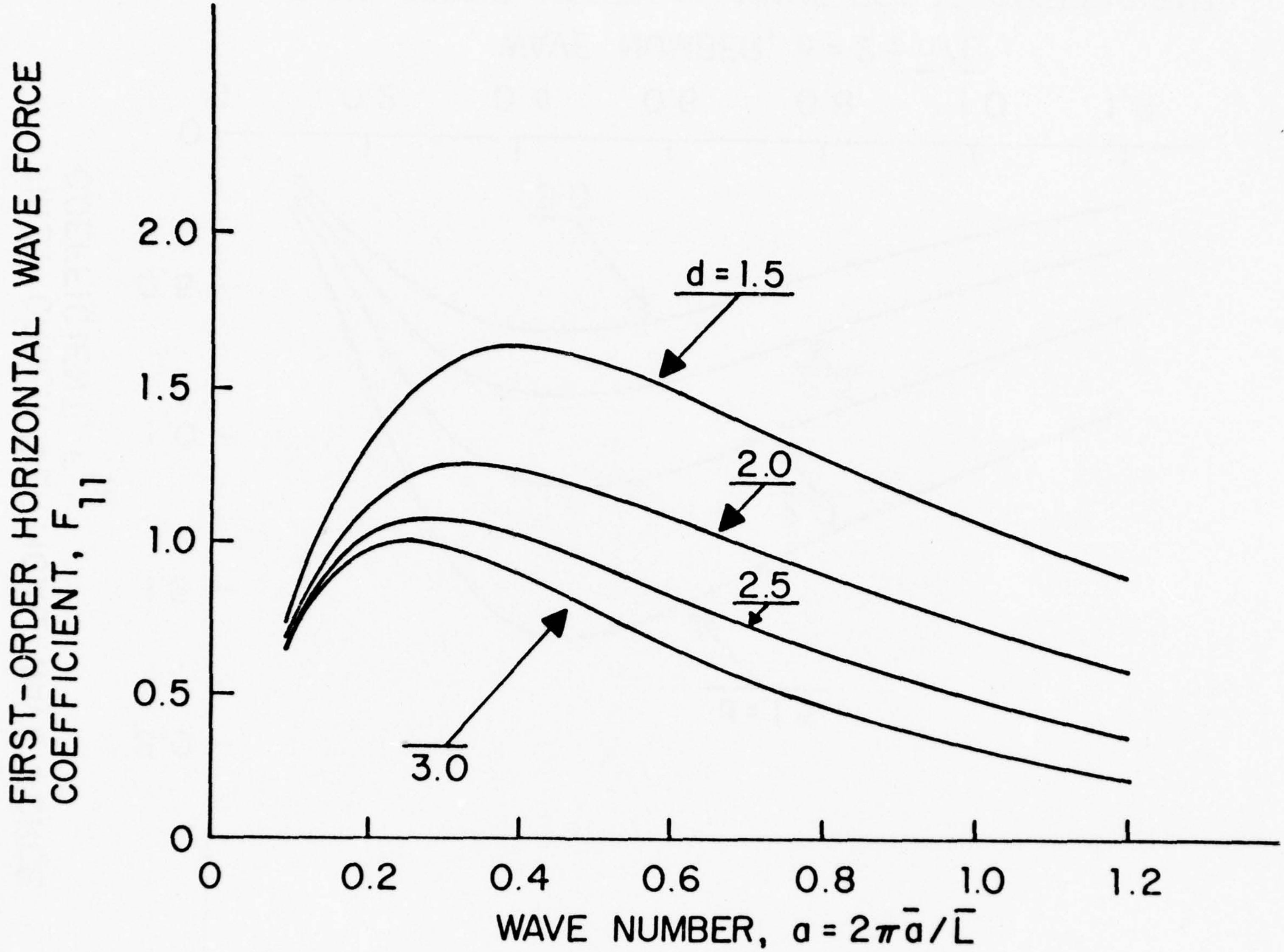

Figure 3. FIRST-ORDER HORIZONTAL WAVE FORCE COEFFICIENT: h = 5.0

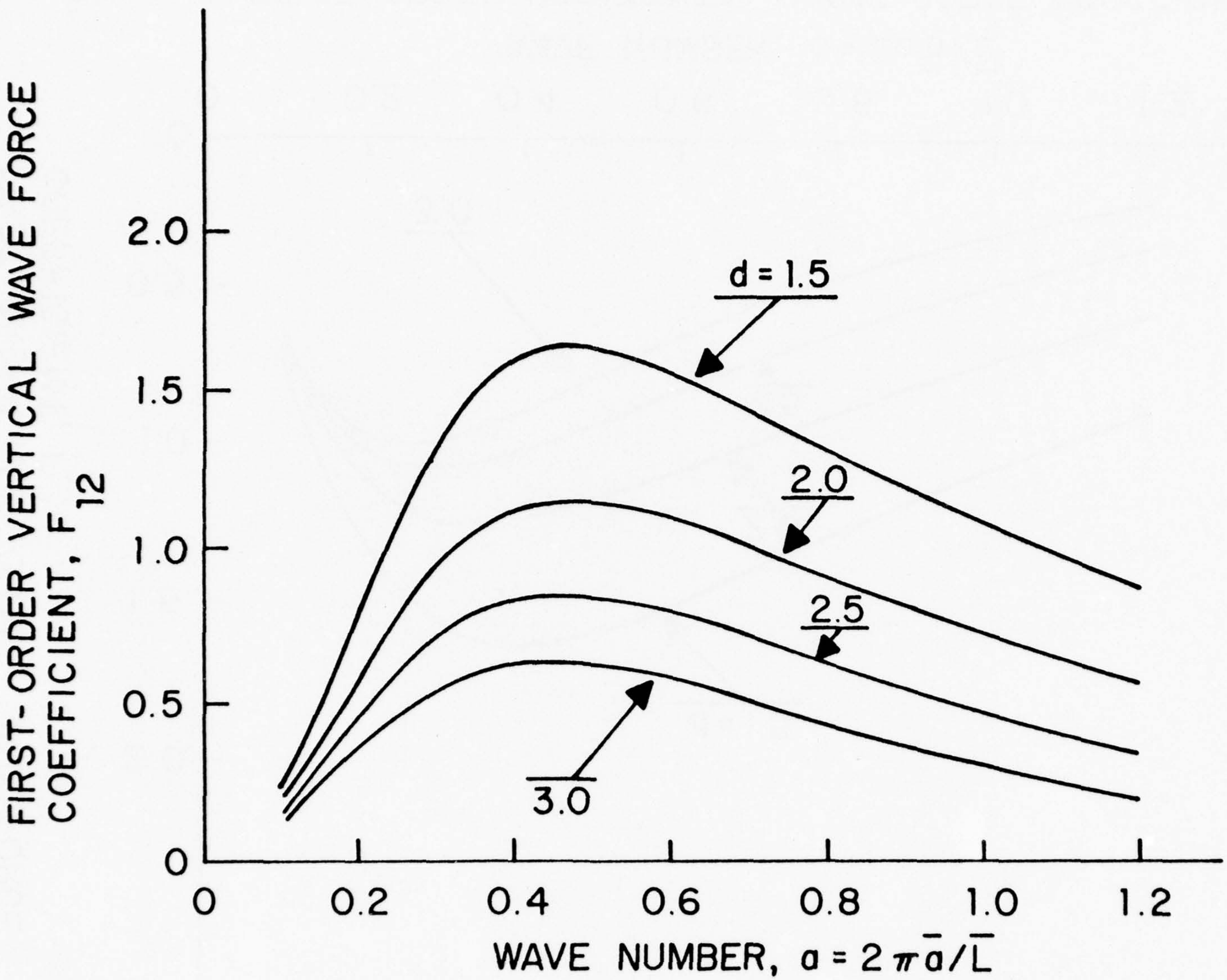

Figure 4. FIRST-ORDER VERTICAL WAVE FORCE COEFFICIENT: $h = 5.0$

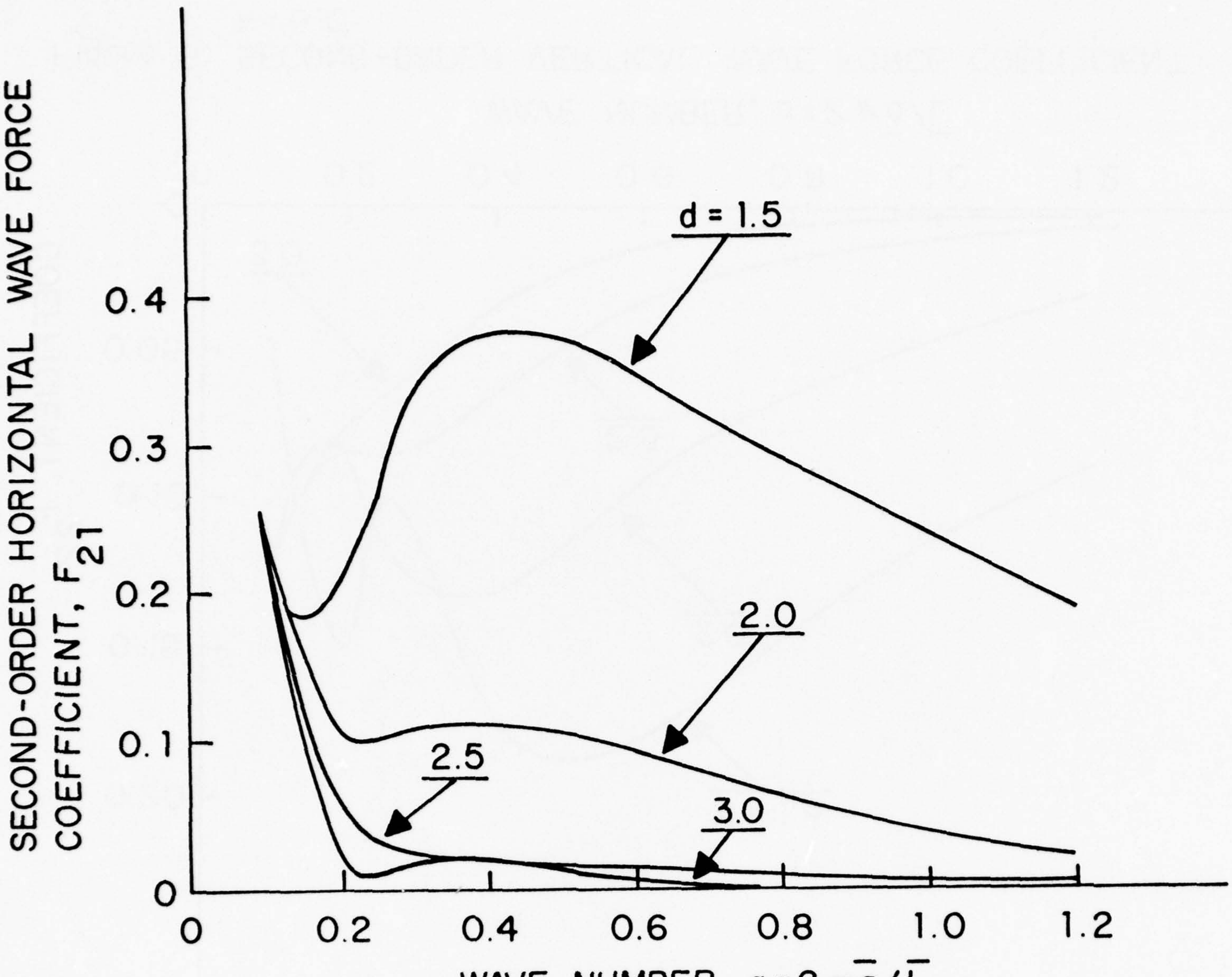

Figure 5. SECOND-ORDER HORIZONTAL WAVE FORCE COEFFICIENT: h = 5.0

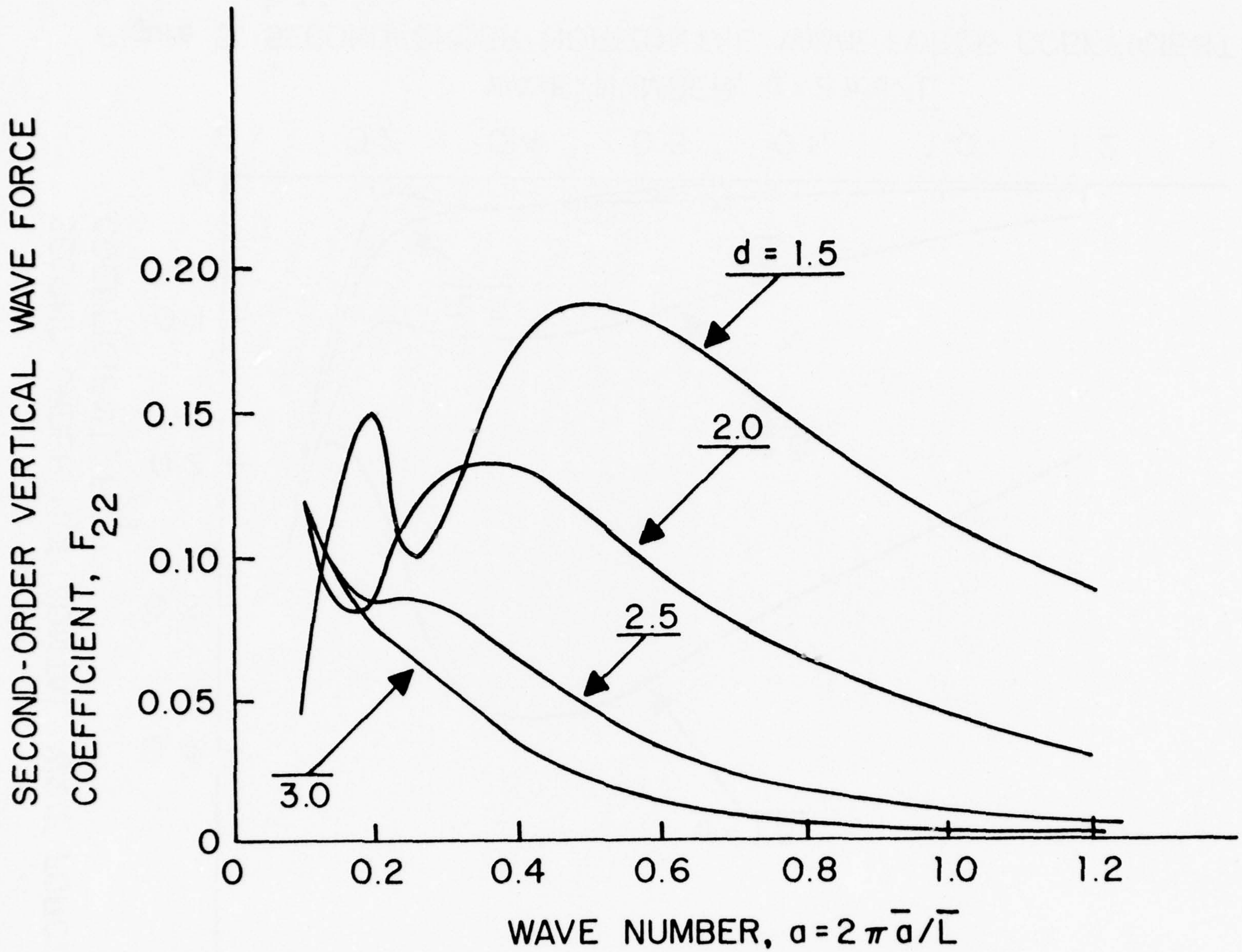

Figure 6. SECOND-ORDER VERTICAL WAVE FORCE COEFFICIENT: h = 5.0

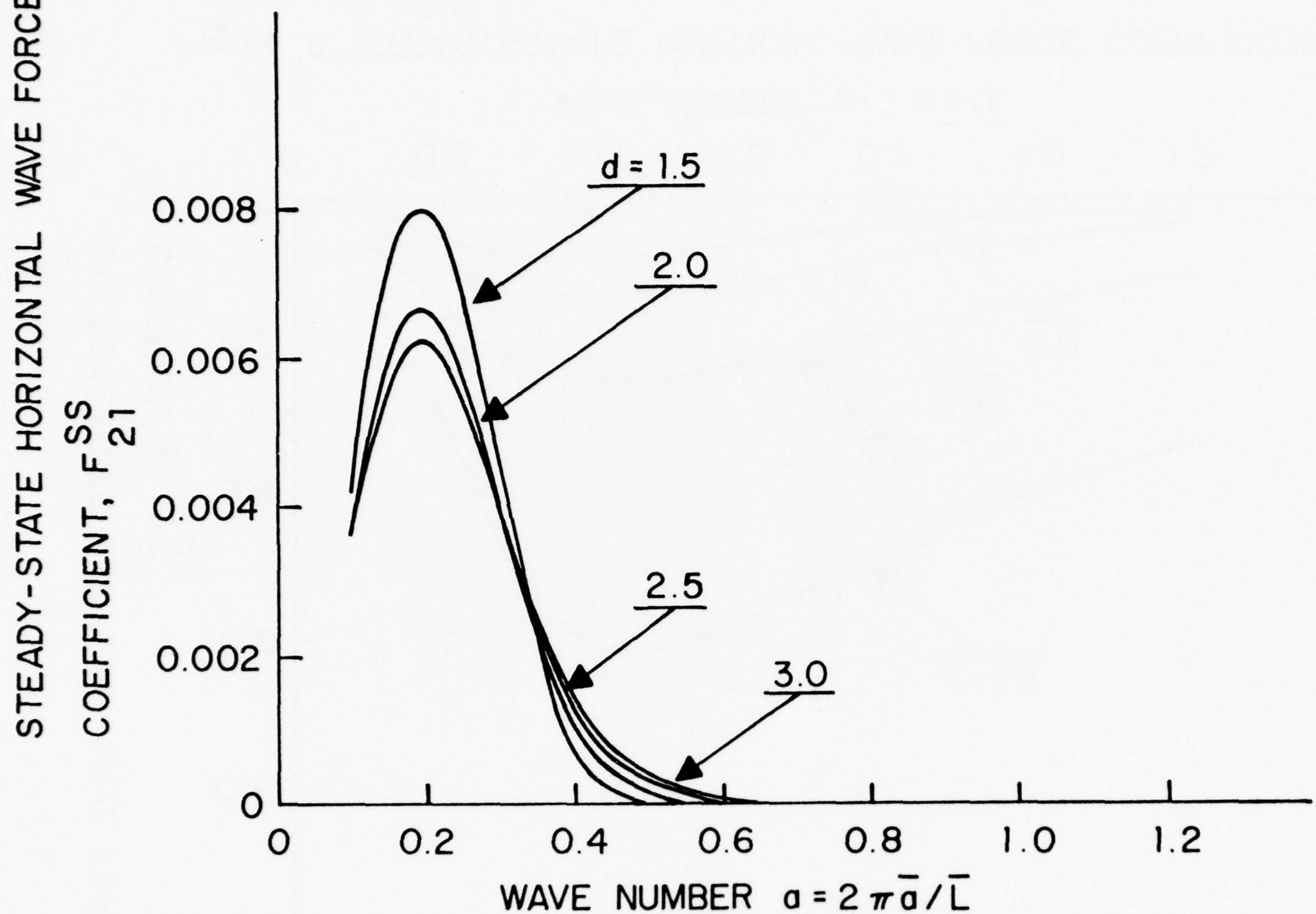

Figure 7. STEADY-STATE HORIZONTAL WAVE FORCE COEFFICIENT: h = 5.0

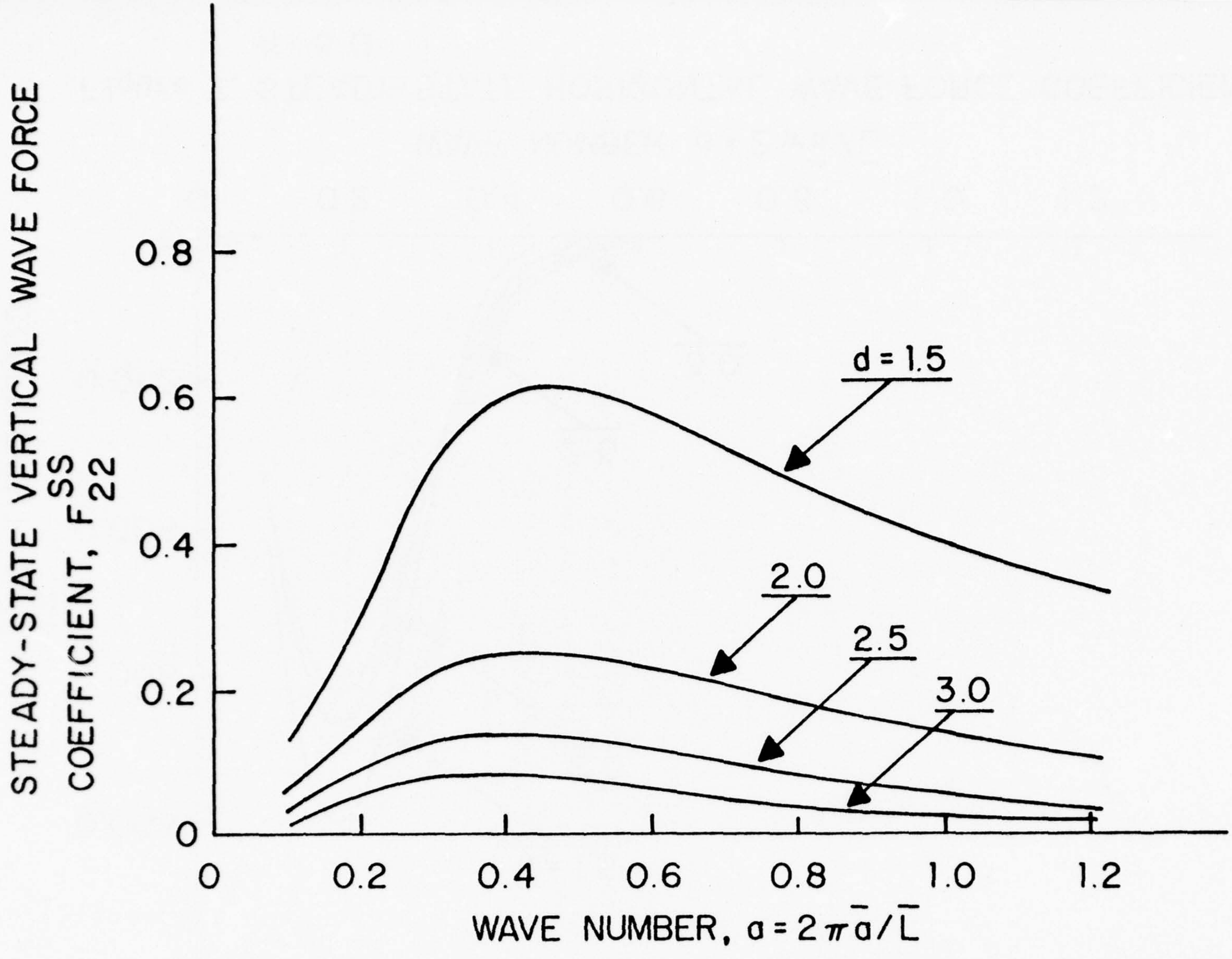

Figure 8. STEADY-STATE VERTICAL WAVE FORCE COEFFICIENT: h = 5.0

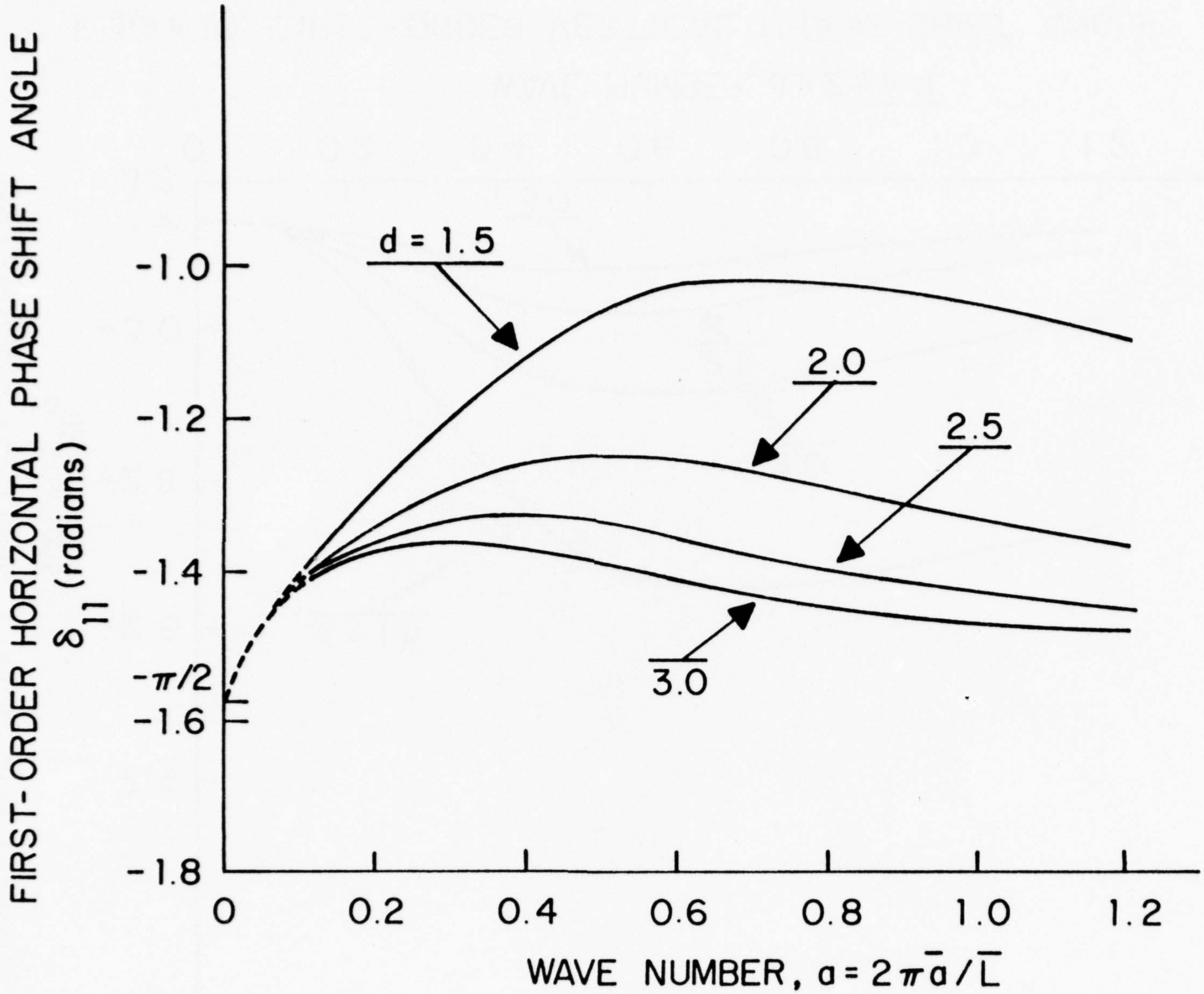

Figure 9. FIRST-ORDER HORIZONTAL PHASE SHIFT ANGLE: h = 5.0

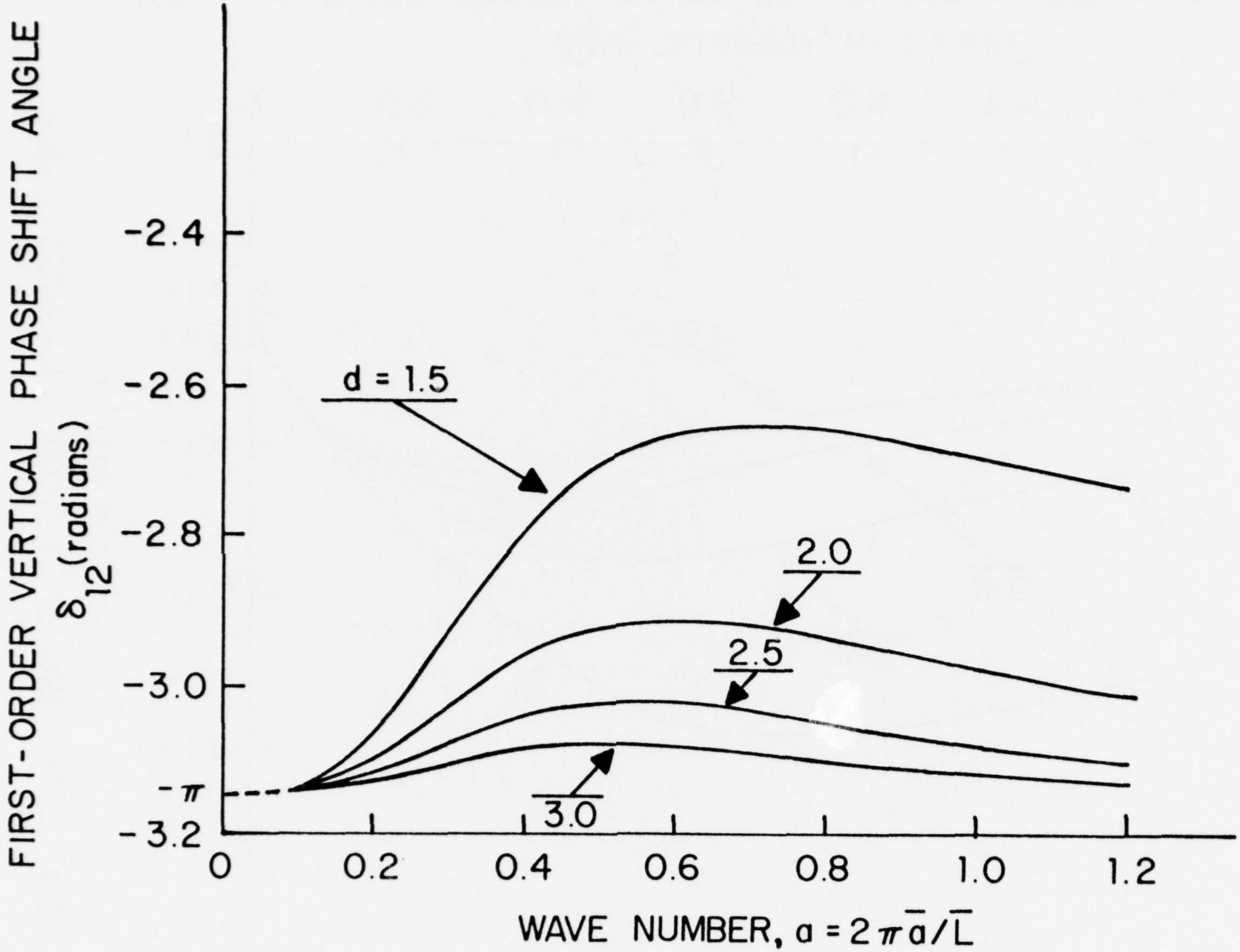

Figure 10. FIRST-ORDER VERTICAL PHASE SHIFT ANGLE: $h = 5.0$

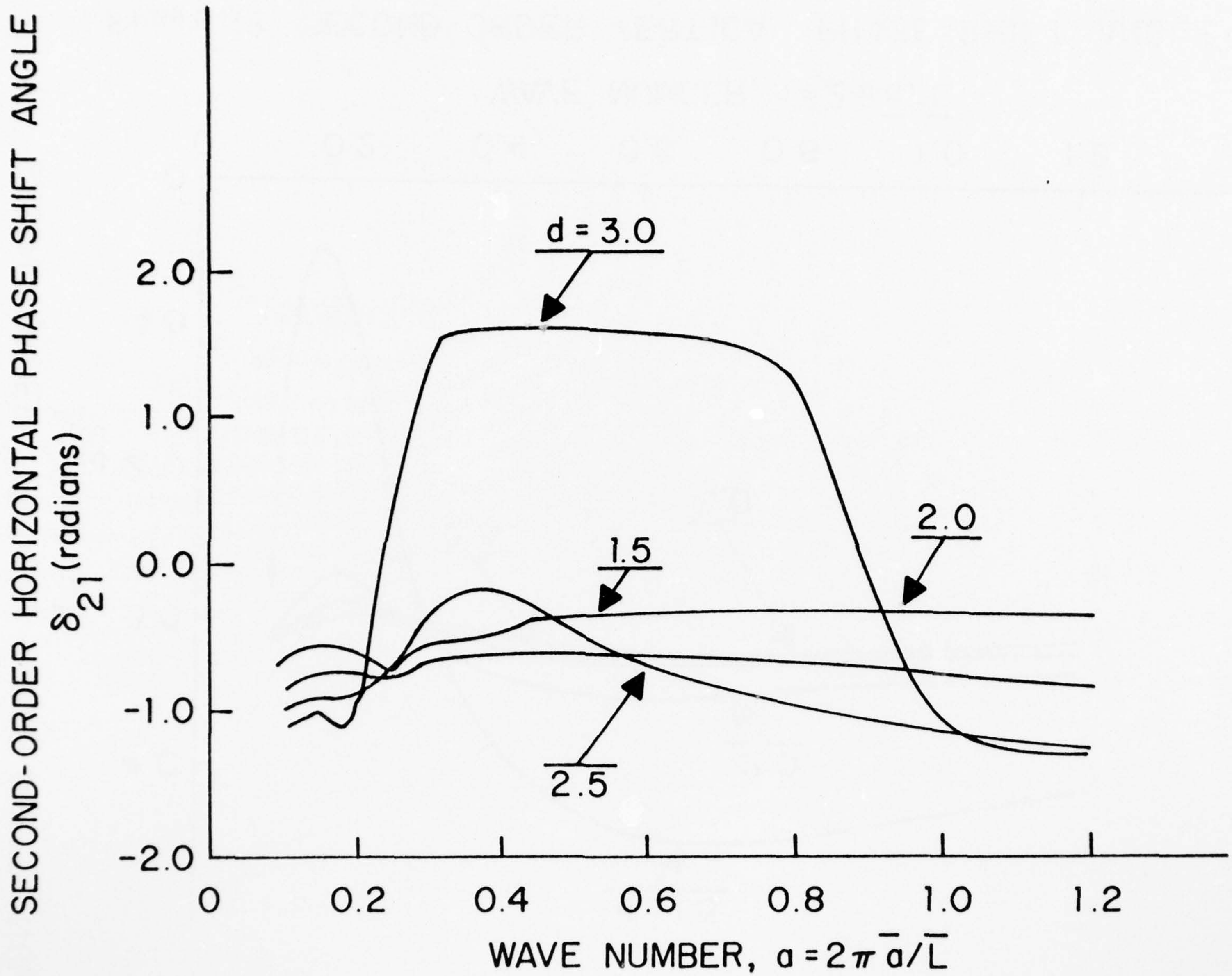

Figure 11. SECOND-ORDER HORIZONTAL PHASE SHIFT ANGLE: h = 5.0

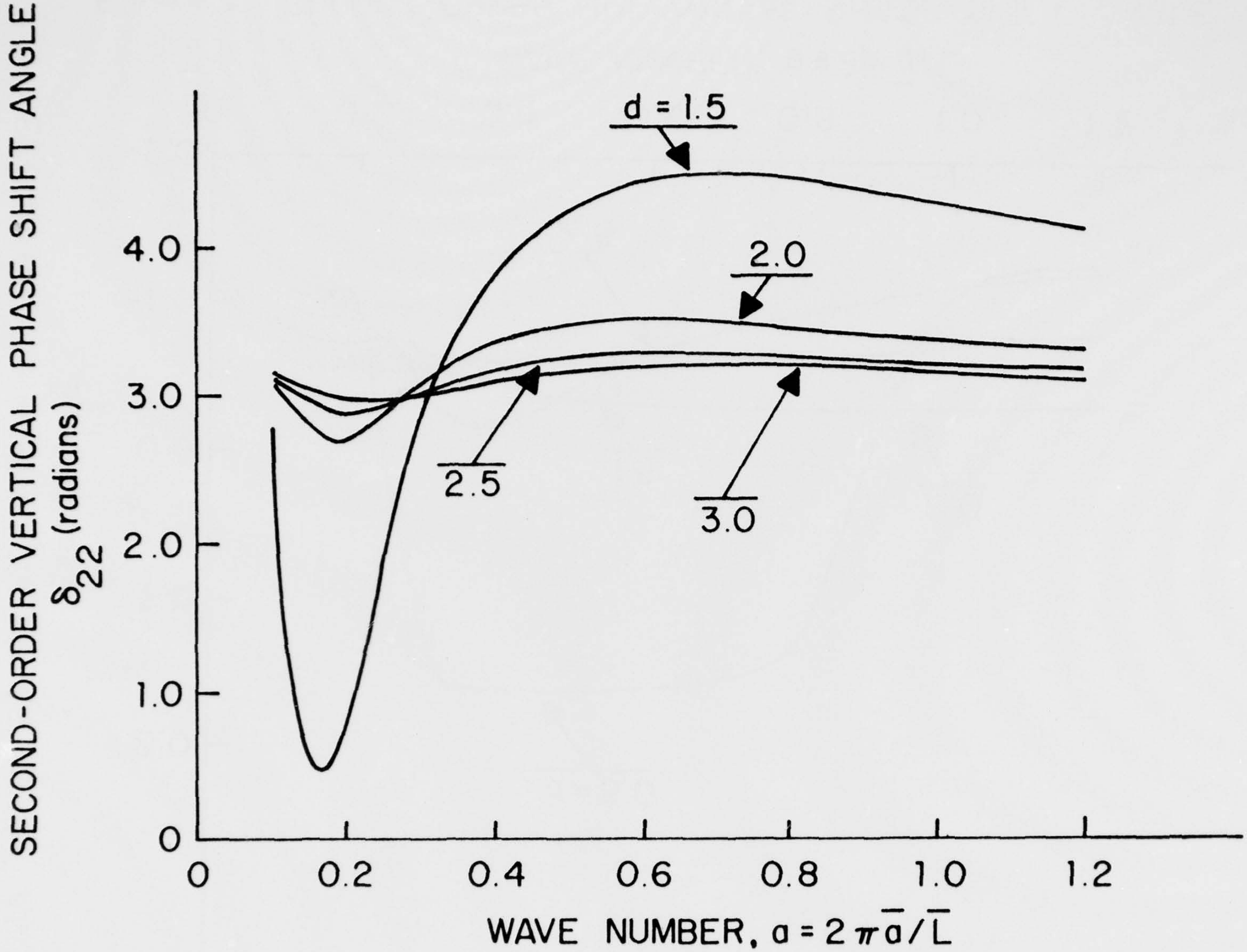

Figure 12. SECOND-ORDER VERTICAL PHASE SHIFT ANGLE: d = 5.0

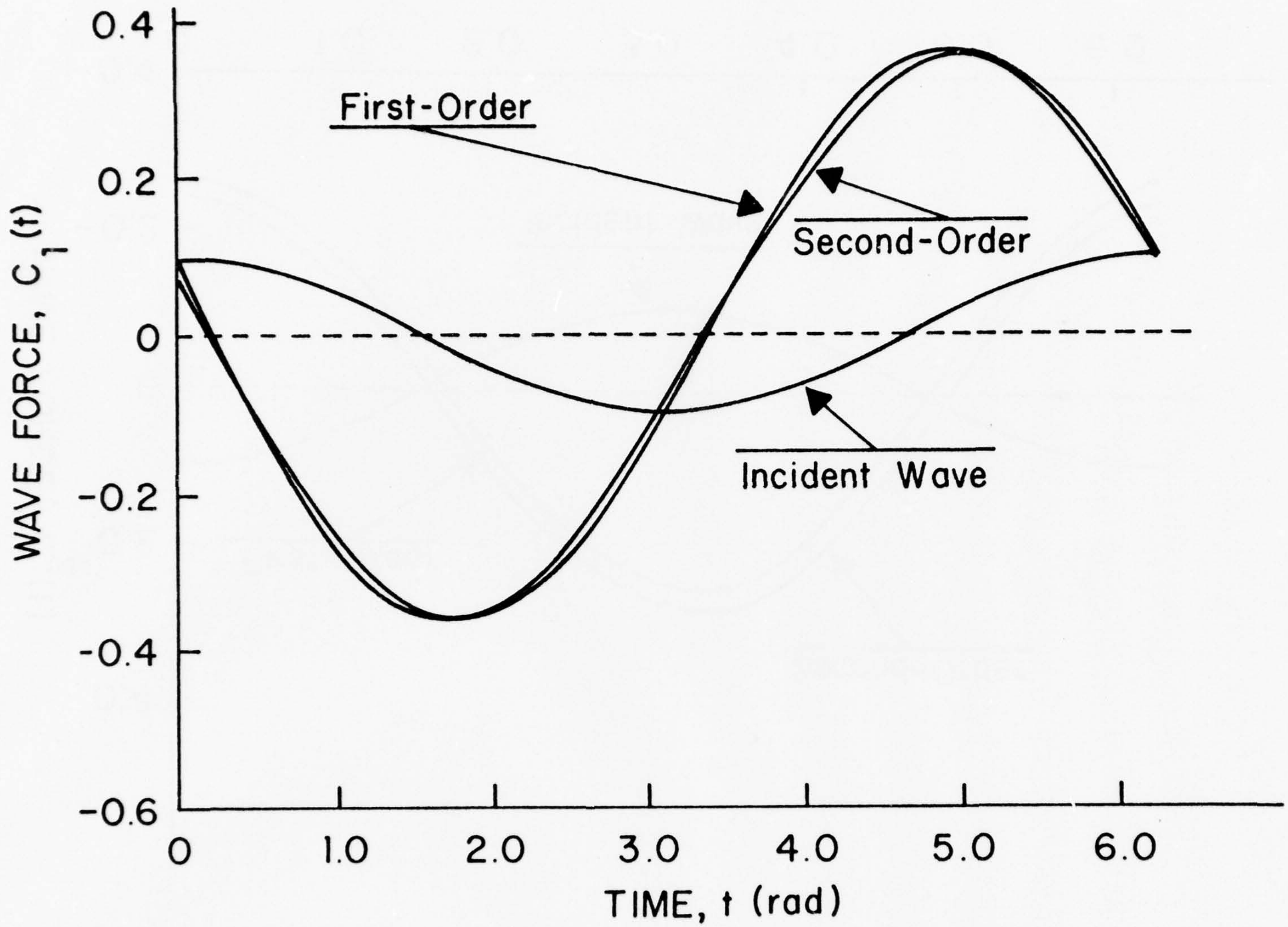

Figure 13. HORIZONTAL WAVE FORCE: a=0.25, h=5.0, d=1.5, H=0.5

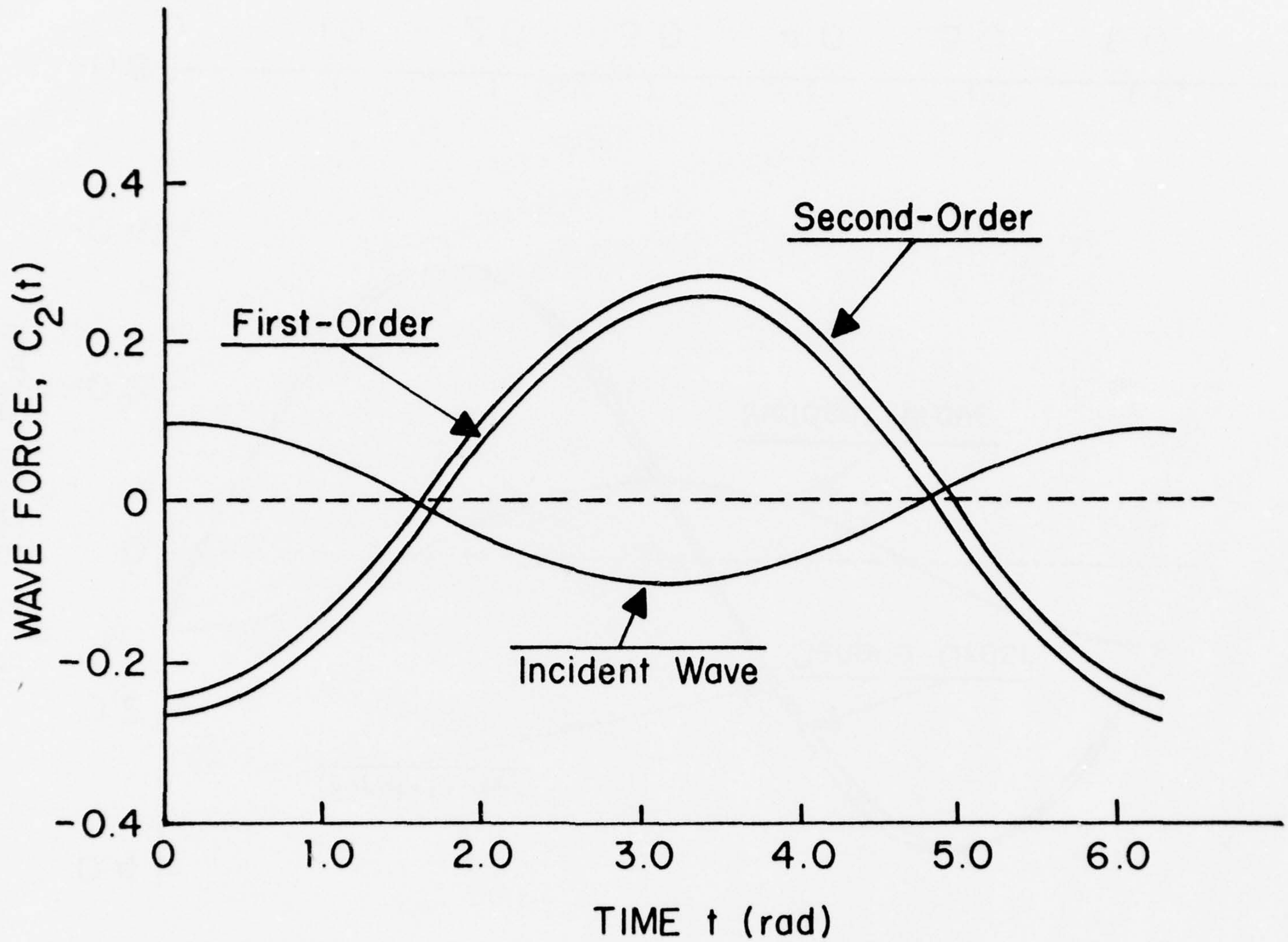

Figure 14. VERTICAL WAVE FORCE: a=0.25, h=5.0, d=1.5, H=0.5

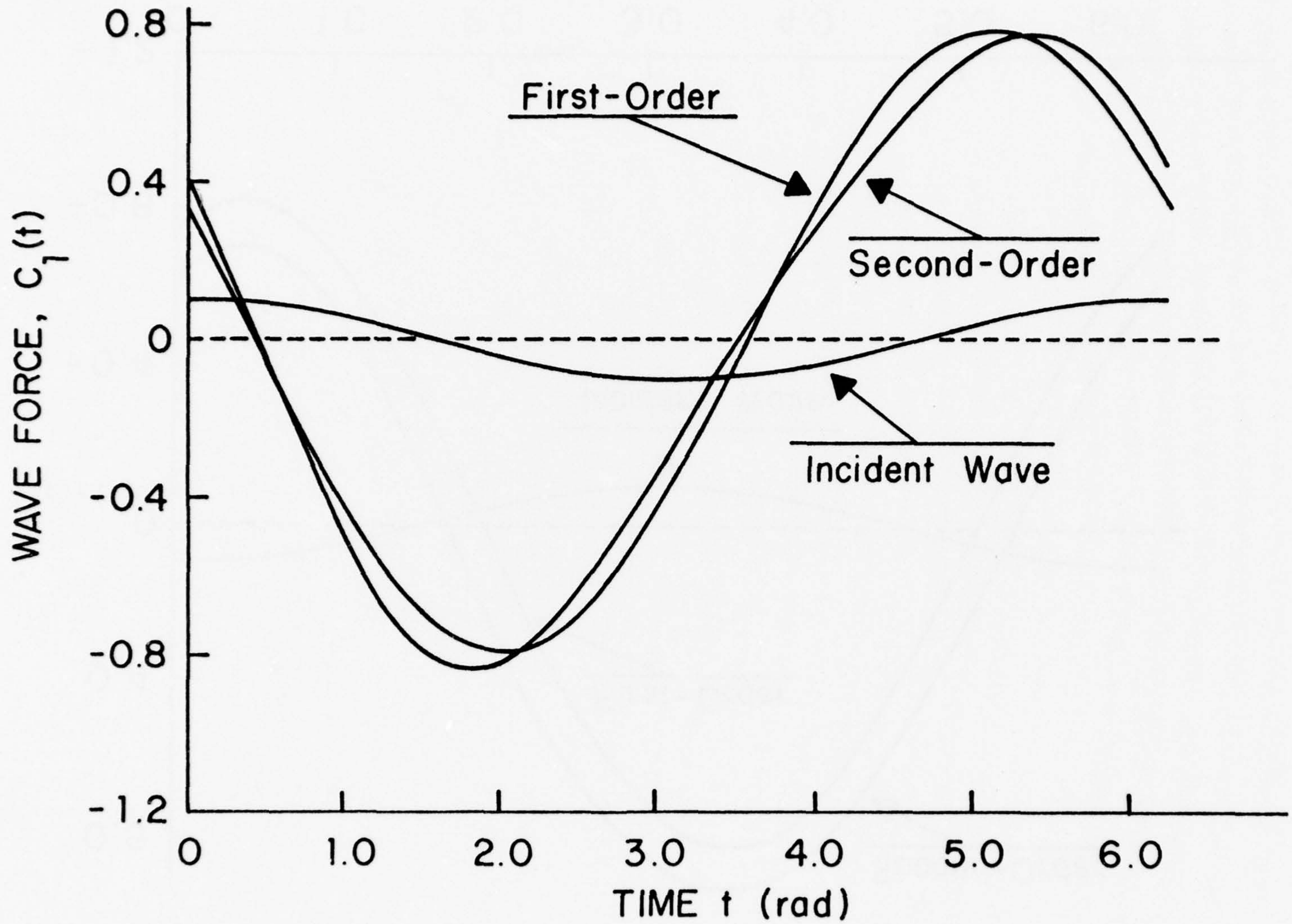

Figure 15. HORIZONTAL WAVE FORCE: a = 0.5, h = 5.0, d = 1.5, H = 0.5

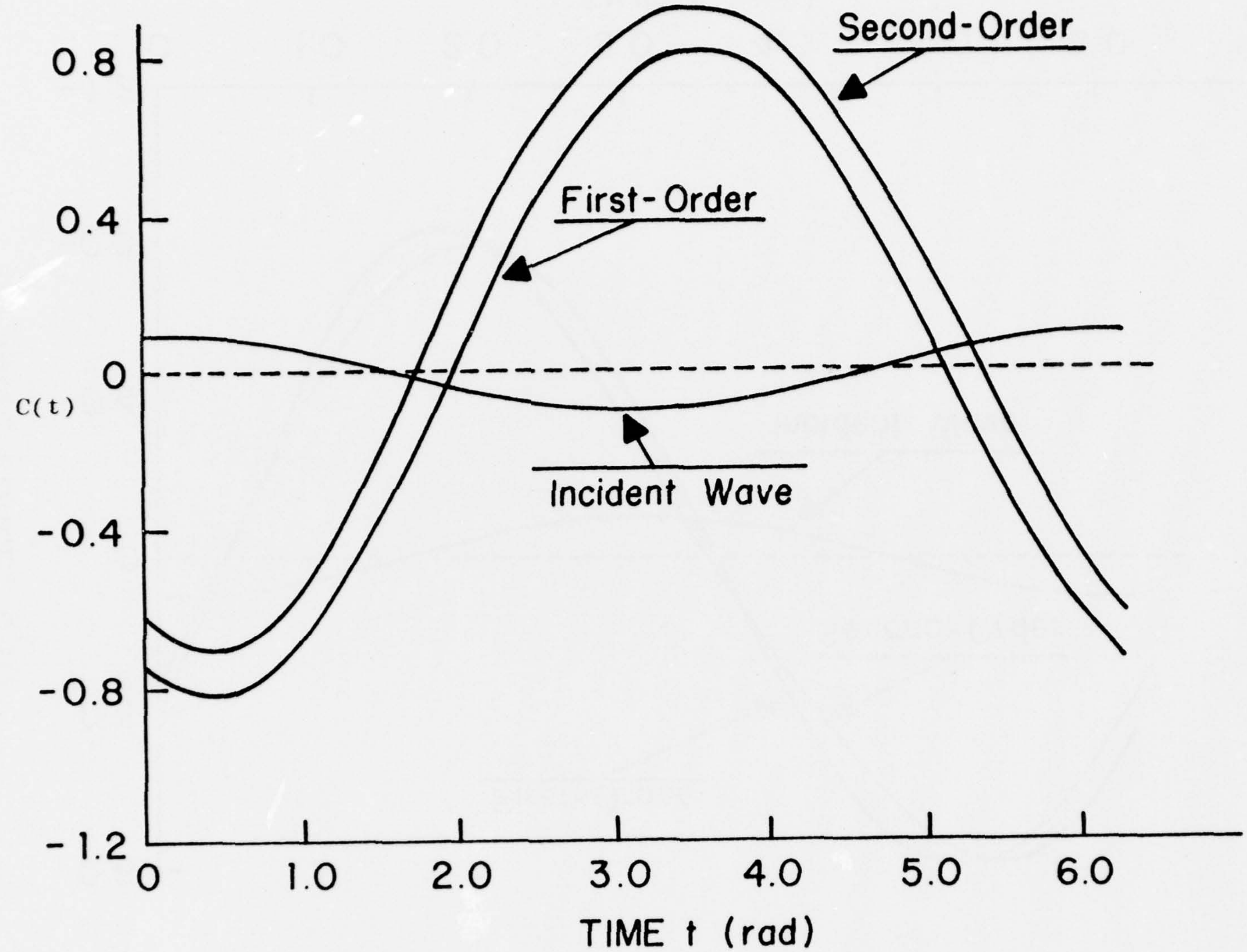

Figure 16. VERTICAL WAVE FORCE: a = 0.5, h = 5.0, d = 1.5, H = 0.5

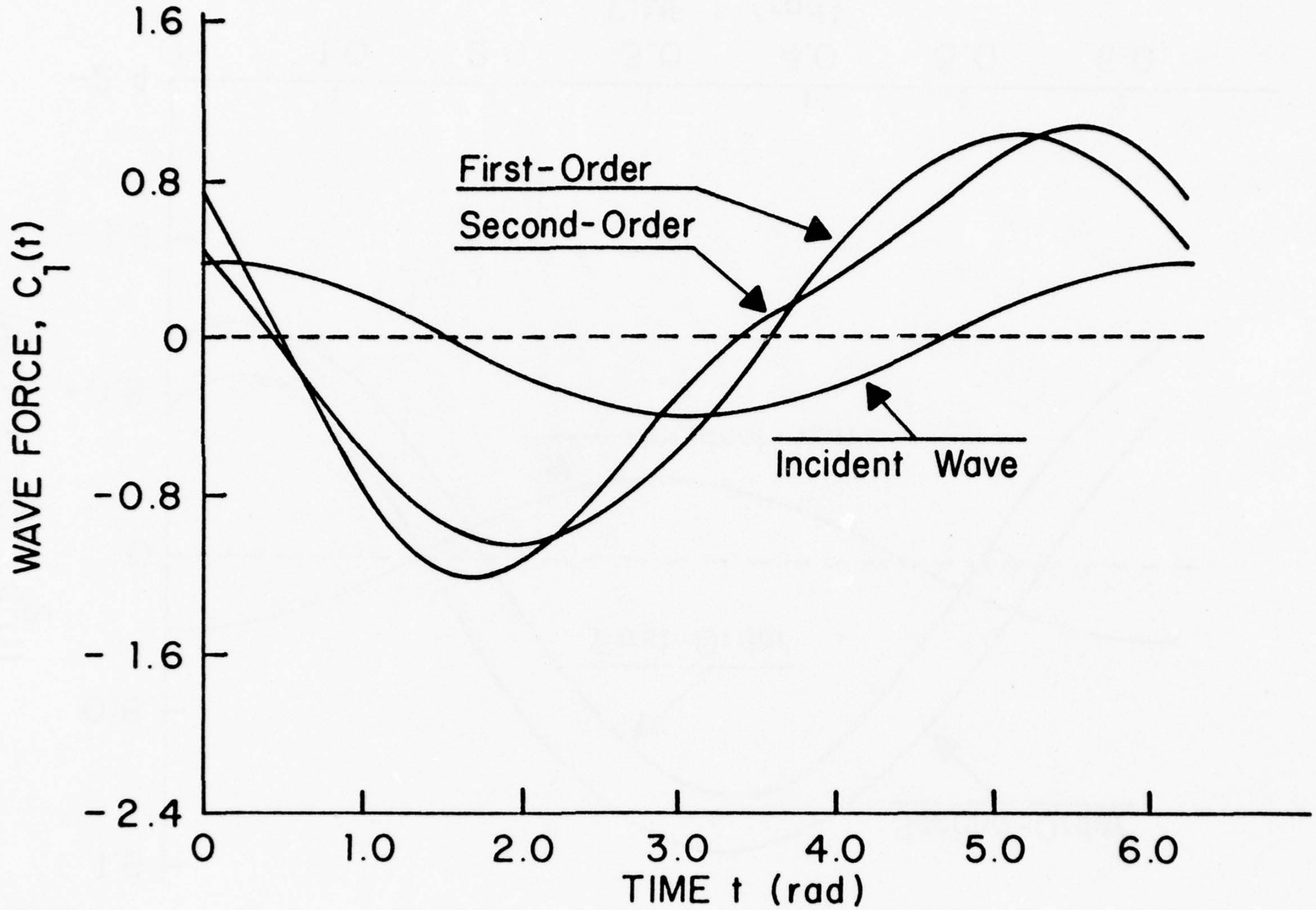

Figure 17. HORIZONTAL WAVE FORCE: $a=1.0$, $h=5.0$, $d=1.5$, $H=0.5$

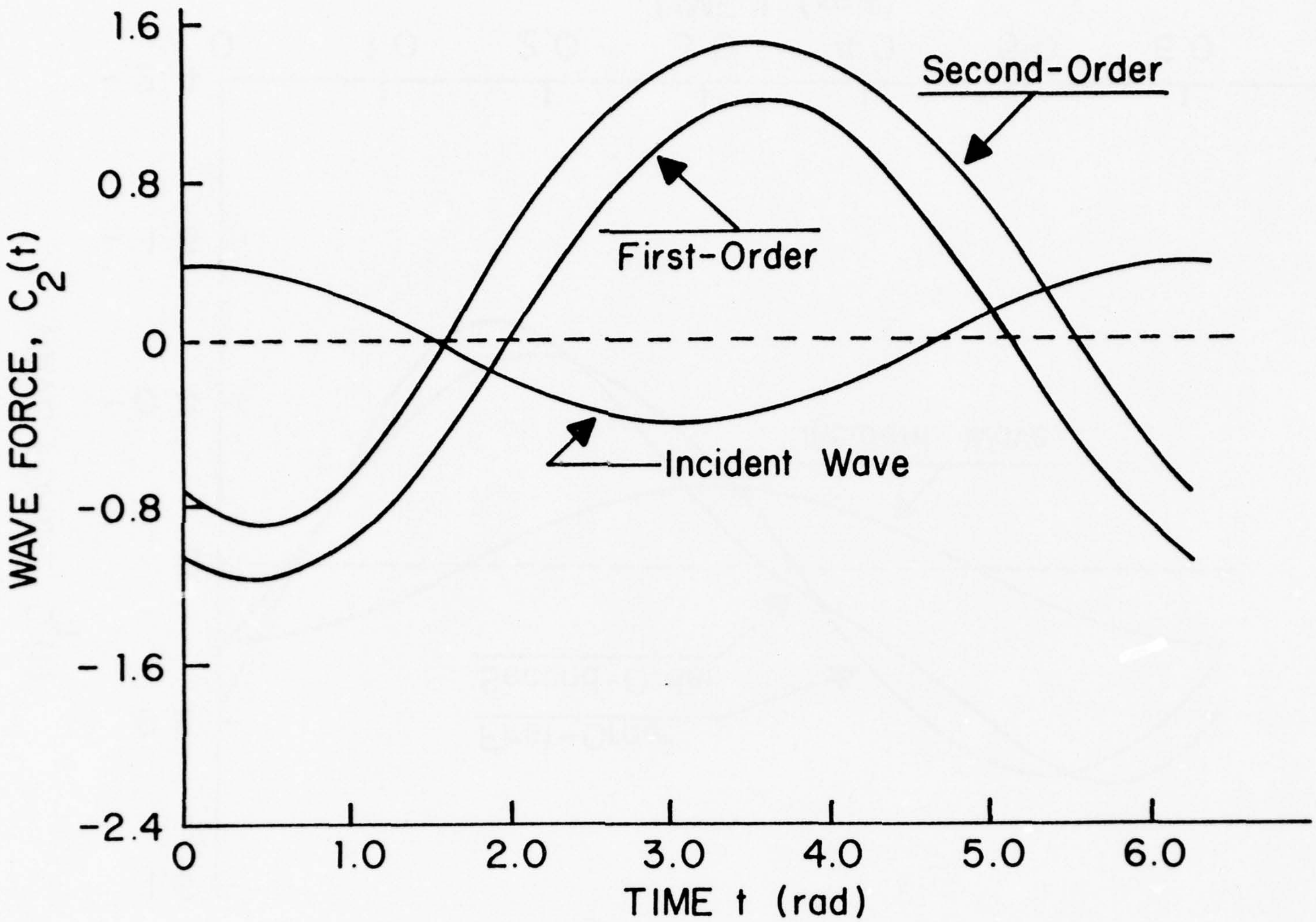

Figure 18. VERTICAL WAVE FORCE: $a = 1.0$, $h = 5.0$, $d = 1.5$, $H = 0.5$

V. CONCLUSIONS

A computer program has been developed to carry out the numerical solution of the second-order wave — cylinder interaction problem. The first-order, second-order, and steady-state force coefficients were determined for the submerged horizontal cylinder.

An approximate method for dealing with the second-order, nonhomogeneous free surface boundary condition was developed which appears to converge except at small values of a, i.e., at very large wave lengths.

APPENDIX A
COMPUTER PROGRAM LISTING

```
      THIS PROGRAM CALCULATES WAVE FORCES AND THEIR PHASE
      SHIFT ANGLES FOR WAVE INTERACTION WITH A SUBMERGED
      HORIZONTAL CYLINDER
      A = 2*PI*CYLINDER RADIUS/WAVE LENGTH = SIGMA**2*
      CYLINDER RADIUS/ACCELERATION OF GRAVITY
      H = MEAN WATER DEPTH/CYLINDER RADIUS
      D = CYLINDER DEPTH/CYLINDER RADIUS
      NPTS = NUMBER OF CYLINDER SURFACE ELEMENTS AND NODAL
      POINTS FOR NUMERICAL EVALUATION
      NSPTS = NUMBER OF FREE SURFACE ELEMENTS AND NODAL
      POINTS FOR NUMERICAL EVALUATION


      COMPLEX GIJ,GNIJ,GNJI,GXIJ,GXJI,GYIJ,GYJI,GIJEXT,GYY,
     1DET,SUM1,SUM2,SUM3,SUM4,U1IS,U1ISX,U1ISY,U1ISYY,
     2U1SS,U1SSX,U1SSY,U1SSYY
      COMPLEX ALPHA(24,24),BETA(24,24),BETAX(24,24),
     1BETAY(24,24),HH(24,1),F(24,1),PK(24,1),F1(24,1),
     2FS(500),U1(24),U1X(24),U1Y(24),U2(24),U2SC1(24),
     3U2SCN1(24),U2SC0(24)
      DIMENSION X(24),Y(24),ANX(24),ANY(24),CHY(25),CHY2(25)
     1,SHY(25),SHY2(25),COEFG(200),COEFG2(200),
     2COSAMU(200,25),COSAM2(200,25),SINAMU(200,25),
     3SINAM2(200,25),AMU(200),AMU4(200),SH2Y(24),CH2Y(24)
      DIMENSION C1(2),C2(2),C3(2),PHASE1(2),PHASE2(2)
      COMMON/CPX/HH,PK
      COMMON/VAR/X,Y,ANX,ANY
      COMMON/GSHY/CHY,CHY2,SHY,SHY2,SH2Y,CH2Y
      COMMON/GMU/COSAMU,COSAM2,SINAMU,SINAM2,AMU,AMU4,COEFG,
     1COEFG2
      COMMON/CONST/H,D,DELTHE,SMIN,NPTS,NSPTS
      COMMON/CPI/PI
      EQUIVALENCE (HH(1),F(1))
      EQUIVALENCE (PK(1),F1(1))
      PI=3.141592


      READ INPUT DATA AND CALCULATE REQUIRED REPEATING
      DATA AND ARRAYS


      CALL GEODAT (A,A2,ANU,ANU4,SH2AH,SH2AH2,SHAH,SHAH2,
     1CHAH,CHAH2,A0,AA,BB,CC,DD,A02,AA2,BB2,CC2,DD2)

      DO 100 I=1,NPTS
      DO 100 J=1,I
      XV=X(I)
      YV=Y(I)
      XC=X(J)
      YC=Y(J)
      ANXI=ANX(I)
      ANYI=ANY(I)
      ANXJ=ANX(J)
      ANYJ=ANY(J)
      IF(ABS(X(I)-X(J)).LT.SMIN) GO TO 50
      SHYI=SHY(I)
      CHYI=CHY(I)
      SHYJ=SHY(J)
      CHYJ=CHY(J)

      CALL GREENS (A,ANU,SH2AH,SHAH,CHAH,COSAMU,SINAMU,AMU,
     1COEFG,SHYI,CHYI,SHYJ,CHYJ,I,J,XV,YV,XC,YC,ANXI,ANYI,
     2ANXJ,ANYJ,GIJ,GNIJ,GNJI,GXIJ,GXJI,GYIJ,GYJI,GYY)

      GO TO 90
   50 CONTINUE

      CALL GREEN (A,ANU,SH2AH,SHAH,CHAH,A0,AA,BB,CC,DD,I,J,
```

```
     1XV,YV,XC,YC,ANXI,ANYI,GIJ,GNIJ,GXIJ,GYIJ,GYY)
      IF(I.EQ.J) GNJI=GNIJ
      IF (I.EQ.J) GXJI=GXIJ
      IF (I.EQ.J) GYJI=GYIJ
      IF(I.EQ.J) GO TO 90
      CALL GREEN (A,ANU,SH2AH,SHAH,CHAH,AO,AA,BB,CC,DD,J,I,
     1XC,YC,XV,YV,ANXJ,ANYJ,GIJEXT,GNJI,GXJI,GYJI,GYY)
   90 CONTINUE

      CALCULATE THE FIRST-ORDER ALPHA AND BETA MATRICES

      ALPHA(I,J)=(1./PI)*GNIJ*DELTHE
      ALPHA(J,I)=(1./PI)*GNJI*DELTHE
      BETA(I,J)=(1./(2.*PI))*GIJ*DELTHE
      BETA(J,I)=BETA(I,J)
      BETAX(I,J)=(1./(2.*PI))*GXIJ*DELTHE
      BETAX(J,I)=(1./(2.*PI))*GXJI*DELTHE
      BETAY(I,J)=(1./(2.*PI))*GYIJ*DELTHE
      BETAY(J,I)=(1./(2.*PI))*GYJI*DELTHE
  100 CONTINUE
      DO 120 I=1,NPTS
      ALPHA(I,I)=ALPHA(I,I)+CMPLX(1.0,0.0)
      BETAX(I,I)=BETAX(I,I)+ANX(I)*CMPLX(0.5,0.0)
      BETAY(I,I)=BETAY(I,I)+ANY(I)*CMPLX(0.5,0.0)
  120 CONTINUE

      GENERATION OF THE FIRST-ORDER DISTRIBUTION FUNCTION,
      F(I,1), BY INVERSION OF ALPHA(I,J)*F(I,1) = HH(I,1)

      CALL COMAT (24,1,ALPHA,HH,DET,INDICA)

      THE HH(I,1) MATRIX IS REPLACED BY THE DISTRIBUTION
      FUNCTION,F(I,1)

      COMPUTATION OF THE FIRST-ORDER SCATTERING POTENTIAL
      FUNCTION, U1SC(I), AND ITS X AND Y PARTIAL DERIVATIVES
      U1SCX(I) AND U1SCY(I), AND COMBINATION WITH THE
      INCIDENT POTENTIAL COUNTERPARTS TO COMPUTE THE TOTAL
      POTENTIAL FUNCTION AND ITS X AND Y DERIVATIVES

      DO 155 I=1,NPTS
      SUM1=(0.0,0.0)
      SUM2=(0.0,0.0)
      SUM3=(0.0,0.0)
      DO 150 J=1,NPTS
      SUM1=SUM1+F(J,1)*BETA(I,J)
      SUM2=SUM2+F(J,1)*BETAX(I,J)
      SUM3=SUM3+F(J,1)*BETAY(I,J)
  150 CONTINUE
      U1(I)=SUM1-(CHY(I)/(A*CHAH))*CEXP(CMPLX(0.0,A*X(I)))
      U1X(I)=SUM2-CMPLX(0.0,1.0)*CHY(I)*CEXP(CMPLX(0.0,
     1A*X(I)))/CHAH
      U1Y(I)=SUM3-SHY(I)*CEXP(CMPLX(0.0,A*X(I)))/CHAH
  155 CONTINUE

      EVALUATION OF THE FREE SURFACE PRESSURE DISTRIBUTION
      SOURCE STRENGTH FUNCTION, FS(L)

      DELX=0.015625*PI/A
      WRITE (6,180) DELX
  180 FORMAT (5X//5X,'DELX=',F10.5///)
```

```
      SP=0.0
      CST=.5*DELTHE/PI
      ICOUNT=1
      DO 250 L=1,NSPTS
      SUM1=(0.0,0.0)
      SUM2=(0.0,0.0)
      SUM3=(0.0,0.0)
      SUM4=(0.0,0.0)
      DO 245 I=1,NPTS
      XV=SP
      YV=0.0
      XC=X(I)
      YC=Y(I)
      ANXI=0.0
      ANYI=1.0
      ANXJ=ANX(I)
      ANYJ=ANY(I)
      SHYI=SHY(25)
      CHYI=CHY(25)
      SHYJ=SHY(I)
      CHYJ=CHY(I)
      IF (ABS(XV-XC).LE.SMIN) GO TO 240

      CALL GREENS (A,ANU,SH2AH,SHAH,CHAH,COSAMU,SINAMU,AMU,
     1COEFG,SHYI,CHYI,SHYJ,CHYJ,25,I,XV,YV,XC,YC,ANXI,ANYI,
     2ANXJ,ANYJ,GIJ,GNIJ,GNJI,GXIJ,GXJI,GYIJ,GYJI,GYY)

      GO TO 241
  240 CONTINUE

      CALL GREEN (A,ANU,SH2AH,SHAH,CHAH,AO,AA,BB,CC,DD,25,I,
     1XV,YV,XC,YC,ANXI,ANYI,GIJ,GNIJ,GXIJ,GYIJ,GYY)

  241 CONTINUE
      SUM1=SUM1+CST*GIJ*F(I,1)
      SUM2=SUM2+CST*GXIJ*F(I,1)
      SUM3=SUM3+CST*GYIJ*F(I,1)
      SUM4=SUM4+CST*GYY*F(I,1)
  245 CONTINUE
      U1IS=(-1./A)*CEXP(CMPLX(0.0,A*SP))
      U1ISX=CMPLX(0.0,-1.0)*CEXP(CMPLX(0.0,A*SP))
      U1ISY=-TANH(A*H)*CEXP(CMPLX(0.0,A*SP))
      U1ISYY=-A*CEXP(CMPLX(0.0,A*SP))
      U1SS=SUM1
      U1SSX=SUM2
      U1SSY=SUM3
      U1SSYY=SUM4
      FS(L)=(2.*A/(3.*ANU))*(U1IS*U1SSYY+U1ISYY*U1SS+U1SS*
     1U1SSYY-6.*U1ISY*U1SSY-3.*U1SSY*U1SSY-4.*U1ISX*U1SSX
     2-2.*U1SSX*U1SSX)
      IF (ICOUNT.EQ.0) GO TO 248
      SP=FLOAT(L+1)*DELX/2.
      ICOUNT=0
      GO TO 250
  248 SP=-SP
      ICOUNT=1
  250 CONTINUE

      CALCULATION OF THE PARTICULAR SOLUTION PORTION OF THE
      SECOND-ORDER SCATTERING POTENTIAL AND ITS NORMAL
      DERIVATIVE, U2SC1(I) AND U2SCN1(I)

      CST=.5*DELX/PI
      DO 350 I=1,NPTS
      SUM1=(0.0,0.0)
      SUM2=(0.0,0.0)
      SP=0.0
      ICOUNT=1
      DO 345 L=1,NSPTS
      XV=X(I)
```

```
      YV=Y(I)
      XS=SP
      YS=0.0
      ANXI=ANX(I)
      ANYI=ANY(I)
      ANXJ=0.0
      ANYJ=1.0
      SHYI=SHY(I)
      CHYI=CHY(I)
      SHYJ=SHY(25)
      CHYJ=CHY(25)
      IF (ABS(XV-XS).LE.SMIN) GO TO 340

      CALL GREENS (A2,ANU4,SH2AH2,SHAH2,CHAH2,COSAM2,SINAM2,
     1AMU4,COEFG2,SHYI,CHYI,SHYJ,CHYJ,I,25,XV,YV,XS,YS,
     2ANXI,ANYI,ANXJ,ANYJ,GIJ,GNIJ,GNJI,GXIJ,GXJI,GYIJ,GYJI,
     3GYY)

      GO TO 341
  340 CONTINUE

      CALL GREEN (A2,ANU4,SH2AH2,SHAH2,CHAH2,AO2,AA2,BB2,CC2
     1,DD2,I,25,XV,YV,XS,YS,ANXI,ANYI,GIJ,GNIJ,GXIJ,GYIJ,
     2GYY)

  341 CONTINUE
      SUM1=SUM1+CST*GIJ*FS(L)
      SUM2=SUM2+CST*GNIJ*FS(L)
      IF (ICOUNT.EQ.0) GO TO 290
      SP=FLOAT(L+1)*DELX/2.
      ICOUNT=0
      GO TO 345
  290 SP=-SP
      ICOUNT=1
  345 CONTINUE
      U2SC1(I)=SUM1
      U2SCN1(I)=SUM2
  350 CONTINUE

      CALCULATION OF THE HOMOGENEOUS SOLUTION PORTION
      OF THE SECOND-ORDER SCATTERING POTENTIAL, U2SCO(I)


      DO 500 I=1,NPTS
      DO 500 J=1,I
      XV=X(I)
      YV=Y(I)
      XC=X(J)
      YC=Y(J)
      ANXI=ANX(I)
      ANYI=ANY(I)
      ANXJ=ANX(J)
      ANYJ=ANY(J)
      IF(ABS(X(I)-X(J)).LT.SMIN) GO TO 450
      SHYI=SHY(I)
      CHYI=CHY(I)
      SHYJ=SHY(J)
      CHYJ=CHY(J)

      CALL GREENS (A2,ANU4,SH2AH2,SHAH2,CHAH2,COSAM2,SINAM2,
     1AMU4,COEFG2,SHYI,CHYI,SHYJ,CHYJ,I,J,XV,YV,XC,YC,ANXI,
     2ANYI,ANXJ,ANYJ,GIJ,GNIJ,GNJI,GXIJ,GXJI,GYIJ,GYJI,GYY)

      GO TO 490
  450 CONTINUE

      CALL GREEN (A2,ANU4,SH2AH2,SHAH2,CHAH2,AO2,AA2,BB2,CC2
     1,DD2,I,J,XV,YV,XC,YC,ANXI,ANYI,GIJ,GNIJ,GXIJ,GYIJ,
     2GYY)

      IF(I.EQ.J) GNJI=GNIJ
```

```
      IF(I.EQ.J) GO TO 490

      CALL GREEN (A2,ANU4,SH2AH2,SHAH2,CHAH2,AO2,AA2,BB2,CC2
     1,DD2,J,I,XC,YC,XV,YV,ANXJ,ANYJ,GIJEXT,GNJI,GXJI,GYJI,
     2GYY)

  490 CONTINUE
      ALPHA(I,J)=(1./PI)*GYIJ*DELTHE
      ALPHA(J,I)=(1./PI)*GYJI*DELTHE
      BETA(I,J)=(1./(2.*PI))*GIJ*DELTHE
      BETA(J,I)=BETA(I,J)
  500 CONTINUE
      DO 510 I=1,NPTS
      PK(I,1)=2*(PK(I,1)-U2SCN1(I))
  510 CONTINUE
      DO 520 I=1,NPTS
      ALPHA(I,I)=ALPHA(I,I)+CMPLX(1.0,0.0)
  520 CONTINUE

      CALL COMAT (24,1,ALPHA,PK,DET,INDICA)

      DO 540 I=1,NPTS
      SUM=(0.0,0.0)
      DO 530 J=1,NPTS
      SUM=SUM+F1(J,1)*BETA(I,J)
  530 CONTINUE
      U2SCO(I)=SUM
  540 CONTINUE


      EVALUATION OF THE TOTAL SECOND-ORDER POTENTIAL, U2(I)


      DO 550 I=1,NPTS
      U2(I)=U2SC1(I)+U2SCO(I)-(CHY(I)/(2.*A*(SHAH**4)))*
     1CEXP(CMPLX(0.0,2.*A*X(I)))
  550 CONTINUE
      WRITE (6,560)
  560 FORMAT (5X//9X,'I',15X,'U1(I)',24X,'U1X(I)',25X,
     1'U1Y(I)',24X,'U2(I)'//)
      DO 580 I=1,NPTS
      WRITE (6,570) I,U1(I),U1X(I),U1Y(I),U2(I)
  570 FORMAT (5X,I5,4(F16.6,F14.6))
  580 CONTINUE


      EVALUATION OF THE FIRST-ORDER, SECOND-ORDER PERIODIC,
      AND SECOND-ORDER STEADY STATE FORCE COEFFICIENTS AND
      THE PERIODIC FORCE PHASE SHIFT ANGLES


      SUM1=(0.0,0.0)
      SUM2=(0.0,0.0)
      SUM3=(0.0,0.0)
      SUM4=(0.0,0.0)
      SUM5=0.0
      SUM6=0.0
      DO 720 I=1,NPTS
      SUM1=SUM1+U1(I)*ANX(I)*DELTHE
      SUM2=SUM2+U1(I)*ANY(I)*DELTHE
      SUM3=SUM3+((6.*ANU*ANU*U2(I)/A)-U1X(I)*U1X(I)-U1Y(I)
     1*U1Y(I))*ANX(I)*DELTHE
      SUM4=SUM4+((6.*ANU*ANU*U2(I)/A)-U1X(I)*U1X(I)-U1Y(I)
     1*U1Y(I))*ANY(I)*DELTHE
      SUM5=SUM5+(CABS(U1X(I)*U1X(I))+CABS(U1Y(I)*U1Y(I))
     1+ANU*ANU/A**2-1.0)*ANX(I)*DELTHE
      SUM6=SUM6+(CABS(U1X(I)*U1X(I))+CABS(U1Y(I)*U1Y(I))
     1+ANU*ANU/A**2-1.0)*ANY(I)*DELTHE
  720 CONTINUE
      C1(1)=CABS(SUM1*A)
      C1(2)=CABS(SUM2*A)
      C2(1)=CABS(SUM3*A*A/(4.*ANU))
```

```
      C2(2)=CABS(SUM4*A*A/(4.*ANU))
      C3(1)=SUM5*A*A/(4.*ANU)
      C3(2)=SUM6*A*A/(4.*ANU)
      PHASE1(1)=ATAN2(AIMAG(SUM1*A),REAL(SUM1*A))
      PHASE1(2)=ATAN2(AIMAG(SUM2*A),REAL(SUM2*A))
      PHASE2(1)=ATAN2(AIMAG(SUM3*A*A/(4.*ANU)),REAL(SUM3*A*A
     1/(4.*ANU)))
      PHASE2(2)=ATAN2(AIMAG(SUM4*A*A/(4.*ANU)),REAL(SUM4*A*A
     1/(4.*ANU)))
      WRITE (6,730) C1(1),C1(2),C2(1),C2(2),C3(1),C3(2),
     1PHASE1(1),PHASE1(2),PHASE2(1),PHASE2(2)
  730 FORMAT (5X///5X,'C1(1)=',F8.5,5X,'C1(2)=',F8.5//5X,
     1'C2(1)=',F8.5,5X,'C2(2)=',F8.5//5X,'C3(1)=',F8.5,5X,
     2'C3(2)=',F8.5//5X,'PHASE1(1)=',F8.5,5X,'PHASE1(2)=',
     3F8.5//5X,'PHASE2(1)=',F8.5,5X,'PHASE2(2)=',F8.5//)
      STOP
      END

      THIS SUBROUTINE READS THE INPUT GEOMETRICAL DATA AND
      CALCULATES THE FIRST 200 ROOTS OF AMU*TAN(AMU*H) +
      ANU = 0 AND AMU4*TAN(AMU4*H) + ANU4 = 0.  IT ALSO
      GENERATES ARRAYS OF CERTAIN FUNCTIONS AND COEFFICIENTS
      USED IN GREEN AND GREENS AS WELL AS THE HH AND PK
      MATRICES

      SUBROUTINE GEODAT (A,A2,ANU,ANU4,SH2AH,SH2AH2,SHAH,
     1SHAH2,CHAH,CHAH2,A0,AA,BB,CC,DD,A02,AA2,BB2,CC2,DD2)

      COMPLEX HH(24,1),PK(24,1)
      DIMENSION X(24),Y(24),ANX(24),ANY(24),CHY(25),CHY2(25)
     1,SHY(25),SHY2(25),COEFG(200),COEFG2(200),AMU(200),
     2AMU4(200),COSAMU(200,25),COSAM2(200,25),SINAMU(200,25)
     3,SINAM2(200,25),SH2Y(24),CH2Y(24),XX(24)
      COMMON/CPX/HH,PK
      COMMON/VAR/X,Y,ANX,ANY
      COMMON/GSHY/CHY,CHY2,SHY,SHY2,SH2Y,CH2Y
      COMMON/GMU/COSAMU,COSAM2,SINAMU,SINAM2,AMU,AMU4,COEFG,
     1COEFG2
      COMMON/CONST/H,D,DELTHE,SMIN,NPTS,NSPTS
      COMMON/CPI/PI
      READ(5,5) A,H,D,SMIN,NPTS,NSPTS
    5 FORMAT (4F10.5,2I4)
      ANU=A*TANH(A*H)
      SH2AH=SINH(2.*A*H)
      CHAH=COSH(A*H)
      SHAH=SINH(A*H)
      A0=TANH(A*H)
      AA=1./CHAH**2
      BB=-H*SHAH/(CHAH**2)
      CC=-H*H*(1.-SHAH**2)/(3.*CHAH**3)
      DD=H**3*SHAH*(2.*CHAH**2+3.*(1.-SHAH**2))/(CHAH**4*12)
      DELTHE=2.*PI/NPTS
      THETA=DELTHE/2.
      DO 15 I=1,NPTS
      X(I)=COS(THETA)
      Y(I)=SIN(THETA)-D
      ANX(I)=X(I)
      ANY(I)=Y(I)+D
      SHY(I)=SINH(A*(H+Y(I)))
      CHY(I)=COSH(A*(H+Y(I)))
      SH2Y(I)=SINH(2.*A*(H+Y(I)))
      CH2Y(I)=COSH(2.*A*(H+Y(I)))
      HH(I,1)=(2./CHAH)*CMPLX(ANY(I)*SHY(I),ANX(I)*CHY(I))*
     1CEXP(CMPLX(0.0, A*X(I)))
      PK(I,1)=(ANY(I)*SH2Y(I)+CMPLX(0.0,1.0)*ANX(I)*CH2Y(I))
     1*CEXP(CMPLX(0.0,2.0*A*X(I)))/SHAH**4
      THETA=THETA+DELTHE
   15 CONTINUE
      SHY(25)=SHAH
      CHY(25)=CHAH
```

```
      B=ANU*H
      DO 50 K=1,200
      XX(1)=PI*K
      DO 25 I=1,20
      II=I
      YY=XX(I)
      XX(I+1)=XX(1)-ATAN2(B,YY)
      IF(ABS((XX(I+1)-XX(I))/XX(I+1)).LT..0001) GO TO 30
   25 CONTINUE
      IF (II.GE.20) GO TO 27
      GO TO 30
   27 WRITE (6,28)
   28 FORMAT (5X,'AMU ROOT DOES NOT CONVERGE'//)
   30 CONTINUE
      AMU(K)=XX(II)/H
      COEFG(K)=2.*PI*(AMU(K)*AMU(K)+ANU*ANU)/(ANU*AMU(K)-H*
     1AMU(K)*(AMU(K)*AMU(K)+ANU*ANU))
      NPTS1=NPTS+1
      DO 40 I=1,NPTS1
      IF (I.EQ.NPTS1) GO TO 35
      COSAMU(K,I)=COS(AMU(K)*(H+Y(I)))
      SINAMU(K,I)=SIN(AMU(K)*(H+Y(I)))
      GO TO 40
   35 COSAMU(K,I)=COS(AMU(K)*H)
      SINAMU(K,I)=SIN(AMU(K)*H)
   40 CONTINUE
   50 CONTINUE
      ANU4=4.*ANU
      B2=ANU4*H
      DO 70 K=1,200
      XX(1)=PI*K
      DO 55 I=1,20
      II=I
      YY=XX(I)
      XX(I+1)=XX(1)-ATAN2(B2,YY)
      IF(ABS((XX(I+1)-XX(I))/XX(I+1)).LT..0001) GO TO 60
   55 CONTINUE
      IF (II.GE.20) GO TO 57
      GO TO 60
   57 WRITE (6,58)
   58 FORMAT (5X,'AMU4 ROOT DOES NOT CONVERGE'//)
   60 CONTINUE
      AMU4(K)=XX(II)/H
      COEFG2(K)=2.*PI*(AMU4(K)*AMU4(K)+ANU4*ANU4)/(ANU4*
     1AMU4(K)-H*AMU4(K)*(AMU4(K)*AMU4(K)+ANU4*ANU4))
      NPTS1=NPTS+1
      DO 65 I=1,NPTS1
      IF (I.EQ.NPTS1) GO TO 68
      COSAM2(K,I)=COS(AMU4(K)*(H+Y(I)))
      SINAM2(K,I)=SIN(AMU4(K)*(H+Y(I)))
      GO TO 65
   68 COSAM2(K,I)=COS(AMU4(K)*H)
      SINAM2(K,I)=SIN(AMU4(K)*H)
   65 CONTINUE
   70 CONTINUE
      XX(1)=ANU4
      DO 80 I=1,20
      II=I
      Y2=XX(I)
      XX(I+1)=4.*ANU/TANH(Y2*H)
      IF (ABS((XX(I+1)-XX(I))/XX(I+1)).LT.0.0001) GO TO 85
   80 CONTINUE
      WRITE (6,82)
   82 FORMAT (5X,'A2 ROOT DOES NOT CONVERGE'//)
   85 CONTINUE
      A2=XX(II)
      SH2AH2=SINH(2.*A2*H)
      CHAH2=COSH(A2*H)
      SHAH2=SINH(A2*H)
      AO2=TANH(A2*H)
      AA2=1./CHAH2**2
      BB2=-H*SHAH2/(CHAH2**2)
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      CC2=-H*H*(1.-SHAH2**2)/(3.*CHAH2**3)
      DD2=(H**3)*SHAH2*(2.*CHAH2**2+3.*(1.-SHAH2**2))/
     1((CHAH2**4)*12.)
      DO 90 I=1,NPTS
      SHY2(I)=SINH(A2*(H+Y(I)))
      CHY2(I)=COSH(A2*(H+Y(I)))
   90 CONTINUE
      WRITE (6,95) A,A2,D,H,ANU,ANU4,NPTS,SMIN,NSPTS
   95 FORMAT (5X///3X,'A=',F10.5,4X,'A2=',F10.5,4X,'D=',
     1F10.5,4X,'H=',F10.5,4X,'ANU=',F10.5,4X,'ANU4=',F10.5/
     24X,'NPTS=',I3,4X,'SMIN=',F10.5,4X,'NSPTS=',I3///)
      RETURN
      END


      THIS SUBROUTINE CALCULATES G AND ITS DERIVATIVES,
      GX,GY,GN, AND GYY, BY USE OF THE INTEGRAL FORM FOR THE
      CASE (X(I) - X(J)) LESS THAN SMIN


      SUBROUTINE GREEN (A,ANU,SH2AH,SHAH,CHAH,AO,AA,BB,CC,DD
     1,I,J,X,Y,XI,ETA,ANX,ANY,G,GN,GX,GY,GYY)

      COMPLEX G,GN
      COMPLEX GX,GY,GYY
      DIMENSION TEST(200),TESTT(100),TESTTT(100),SUMOX(15),
     1SUMOY(15),NNN(15),TESTYY(200)
      COMMON/CONST/H,D,DELTHE,SMIN,NPTS,NSPTS
      COMMON/CPI/PI
      P1(Y,ETA,XX,AMU)=COSH(AMU*(Y+H))*COSH(AMU*(ETA+H))*
     1COS(AMU*XX)/(COSH(AMU*H)**2)
      P2X(Y,ETA,X,XI,AMU)=-AMU*COSH(AMU*(ETA+H))*SIN(AMU*
     1(X-XI))*COSH(AMU*(Y+H))/(COSH(AMU*H)**2)
      P2Y(Y,ETA,X,XI,AMU)=AMU*COSH(AMU*(ETA+H))*SINH(AMU*
     1(Y+H))*COS(AMU*(X-XI))/(COSH(AMU*H)**2)
      P2YY(Y,ETA,X,XI,AMU)=AMU*AMU*COSH(AMU*(Y+H))*COSH(AMU*
     1(ETA+H))*COS(AMU*(X-XI))/(COSH(AMU*H)**2)
      Q1(Y,ETA,XX,AMU)=-P1(  Y,ETA,XX,AMU)*(AMU-A)/(AMU*
     1TANH(AMU*H)-ANU)
      Q2X(Y,ETA,X,XI,AMU)=-P2X(Y,ETA,X,XI,AMU)*(AMU-A)/(AMU*
     1TANH(AMU*H)-ANU)
      Q2Y(Y,ETA,X,XI,AMU)=-P2Y(Y,ETA,X,XI,AMU)*(AMU-A)/(AMU*
     1TANH(AMU*H)-ANU)
      Q2YY(Y,ETA,X,XI,AMU)=-P2YY(Y,ETA,X,XI,AMU)*(AMU-A)/
     1(AMU*TANH(AMU*H)-ANU)
      Q1O(Y,ETA,XX)=-COSH(A*(Y+H))*COSH(A*(ETA+H))*COS(A*XX)
     1*A/(COSH(A*H)**2*((A*A-ANU*ANU)*H+ANU))
      Q2OX(Y,ETA,X,XI)=A*COSH(A*(ETA+H))*SIN(A*(X-XI))*COSH
     1(A*(Y+H))*A/(COSH(A*H)**2*((A*A-ANU*ANU)*H+ANU))
      Q2OY(Y,ETA,X,XI)=-A*COSH(A*(ETA+H))*SINH(A*(Y+H))*COS
     1(A*(X-XI))*A/(COSH(A*H)**2*((A*A-ANU*ANU)*H+ANU))
      Q2OYY(Y,ETA,X,XI)=-A*A*COSH(A*(Y+H))*COSH(A*(ETA+H))*
     1COS(A*(X-XI))*A/(COSH(A*H)**2*((A*A-ANU*ANU)*H+ANU))
      Q1S(Y,ETA,XX,AMU)=-P1(Y,ETA,XX,AMU)/(AO+AMU*H*(AA+BB*
     1(AMU-A)+CC*(AMU-A)**2+DD*(AMU-A)**3))
      Q2XS(Y,ETA,X,XI,AMU)=-P2X(Y,ETA,X,XI,AMU)/(AO+AMU*H*
     1(AA+BB*(AMU-A)+CC*(AMU-A)**2+DD*(AMU-A)**3))
      Q2YS(Y,ETA,X,XI,AMU)=-P2Y(Y,ETA,X,XI,AMU)/(AO+AMU*H*
     1(AA+BB*(AMU-A)+CC*(AMU-A)**2+DD*(AMU-A)**3))
      Q2YYS(Y,ETA,X,XI,AMU)=-P2YY(Y,ETA,X,XI,AMU)/(AO+AMU*H*
     1(AA+BB*(AMU-A)+CC*(AMU-A)**2+DD*(AMU-A)**3))
      FUN1(Y,ETA,XX,AMU)=EXP(-AMU*H)*SINH(AMU*ETA)*SINH
     1(AMU*Y)*COS(AMU*XX)/(AMU*COSH(AMU*H))
      FUN2(Y,ETA,XX)=4.*3.14159*COSH(A*(Y+H))*COSH(A*
     1(ETA+H))*COS(A*XX)/(2.*A*H+SH2AH)
      FUNR(X,Y,XI,ETA,ANX,ANY)=((X-XI)*ANX+(Y+ETA)*ANY)/
     1((X-XI)**2+(Y+ETA)**2)
      FUN3X(Y,ETA,X,XI,AMU)=EXP(-AMU*H)*SINH(AMU*ETA)*SINH
     1(AMU*Y)*SIN(AMU*(X-XI))/COSH(AMU*H)
      FUN3Y(Y,ETA,X,XI,AMU)=-EXP(-AMU*H)*SINH(AMU*ETA)*COSH
     1(AMU*Y)*COS(AMU*(X-XI))/COSH(AMU*H)
      FUN3YY(Y,ETA,X,XI,AMU)=-EXP(-AMU*H)*SINH(AMU*ETA)*
```

```
1SINH(AMU*Y)*AMU*AMU*COS(AMU*(X-XI))/COSH(AMU*H)
 FUN4(Y,ETA,XX,ANX,ANY,SIGN)=4.*3.14159*A*COSH(A*
1(ETA+H))*(ANY*SINH(A*(Y+H))*COS(A*XX)-ANX*COSH(A*
2(Y+H))*SIN(A*XX)*SIGN)/(2.*A*H+SH2AH)

  EVALUATION OF THE FINITE INTEGRAL IN G TO DETERMINE
  THE SIZE OF THE SUBDIVISIONS

  XX=ABS(X-XI)
  IF(X.LT.XI) SIGN=-1.0
  IF(X.GT.XI) SIGN=1.0
  IF(X.EQ.XI) SIGN=0.0
  DO 50 N=1,15
  DELMU=2*A/(6*N+3)
  AMU=0.0
  SUM=0.0
  F0=(Q1(Y,ETA,XX,AMU)-Q10(Y,ETA,XX))/(AMU-A)
  LL=6*N+3
  DO 40 NN=1,LL
  IF(ABS(AMU-A).LT..00001) GO TO 10
  IF(ABS(AMU+DELMU-A).LT..00001) GO TO 10
  IF(ABS(AMU+DELMU/3.-A).LT..00001) GO TO 10
  IF(ABS(AMU+2.*DELMU/3.-A).LT..00001) GO TO 10
  F1=(Q1(Y,ETA,XX,AMU+DELMU/3.)-Q10(Y,ETA,XX))/(AMU+
 1DELMU/3.-A)
  F2=(Q1(Y,ETA,XX,AMU+2.*DELMU/3.)-Q10(Y,ETA,XX))/
 1(AMU+2.*DELMU/3.-A)
  F3=(Q1(Y,ETA,XX,AMU+DELMU)-Q10(Y,ETA,XX))/(AMU+DELMU
 1-A)
  GO TO 30
10 F0=(Q1(Y,ETA,XX,AMU+DELMU)-Q10(Y,ETA,XX))/(AMU+DELMU
 1-A)
  GO TO 40
30 SUM=    (DELMU/8.)*(F0+3.*F1+3.*F2+F3)+SUM
  F0=F3
40 AMU=AMU+DELMU
  TEST(N)=SUM
  IF(N-1) 50,50,45
45 MN=N-1
  MM=6*MN+3
  IF(ABS((TEST(N)-TEST(N-1))/TEST(N)).LT..010) GO TO 60
50 CONTINUE
60 CONTINUE
  PV1F=2.*SUM

  EVALUATION OF THE FINITE INTEGRAL IN GN USING
  2*A/MM SUBDIVISION SIZE

  DELMU=2*A/MM
  AMU=0.0
  SUMX=0.0
  SUMY=0.0
  F0=(Q2X(Y,ETA,X,XI,AMU)-Q20X(Y,ETA,X,XI))/(AMU-A)
  Y0=(Q2Y(Y,ETA,X,XI,AMU)-Q20Y(Y,ETA,X,XI))/(AMU-A)
  DO 80 NN=1,MM
  IF(ABS(AMU-A).LT..00001) GO TO 70
  IF(ABS(AMU+DELMU-A).LT..00001) GO TO 70
  IF(ABS(AMU+DELMU/3.-A).LT..00001) GO TO 70
  IF(ABS(AMU+2.*DELMU/3.-A).LT..00001) GO TO 70
  F1=(Q2X(Y,ETA,X,XI,AMU+DELMU/3.)-Q20X(Y,ETA,X,XI))/
 1(AMU+DELMU/3.-A)
  F2=(Q2X(Y,ETA,X,XI,AMU+2.*DELMU/3.)-Q20X(Y,ETA,X,XI))/
 1(AMU+2.*DELMU/3.-A)
  F3=(Q2X(Y,ETA,X,XI,AMU+DELMU)-Q20X(Y,ETA,X,XI))/
 1(AMU+DELMU-A)
  Y1=(Q2Y(Y,ETA,X,XI,AMU+DELMU/3.)-Q20Y(Y,ETA,X,XI))/
 1(AMU+DELMU/3.-A)
  Y2=(Q2Y(Y,ETA,X,XI,AMU+2.*DELMU/3.)-Q20Y(Y,ETA,X,XI))/
```

```
     1(AMU+2.*DELMU/3.-A)
      Y3=(Q2Y(Y,ETA,X,XI,AMU+DELMU)-Q2OY(Y,ETA,X,XI))/
     1(AMU+DELMU-A)
      GO TO 75
   70 CONTINUE
      F0=(Q2X(Y,ETA,X,XI,AMU+DELMU)-Q2OX(Y,ETA,X,XI))/
     1(AMU+DELMU-A)
      Y0=(Q2Y(Y,ETA,X,XI,AMU+DELMU)-Q2OY(Y,ETA,X,XI))/
     1(AMU+DELMU-A)
      GO TO 80
   75 CONTINUE
      SUMX=SUMX+    (DELMU/8.)*(F0+3.*F1+3.*F2+F3)
      SUMY=SUMY+    (DELMU/8.)*(Y0+3.*Y1+3.*Y2+Y3)
      F0=F3
      Y0=Y3
   80 AMU=AMU+DELMU
      PV2FX=2.*SUMX
      PV2FY=2.*SUMY

      EVALUATION OF THE FINITE INTEGRAL IN GYY USING
      2*A/MM SUBDIVISION SIZE

      DELMU=2*A/MM
      AMU=0.0
      SUMYY=0.0
      YY0=(Q2YY(Y,ETA,X,XI,AMU)-Q2OYY(Y,ETA,X,XI))/(AMU-A)
      DO 350 NN=1,MM
      IF(ABS(AMU-A).LT.0.00001) GO TO 320
      IF (ABS(AMU+DELMU-A).LT.0.00001) GO TO 320
      IF (ABS(AMU+DELMU/3.-A).LT.0.00001) GO TO 320
      IF (ABS(AMU+2.*DELMU/3.-A).LT.0.00001) GO TO 320
      YY1=(Q2YY(Y,ETA,X,XI,AMU+DELMU/3.)-Q2OYY(Y,ETA,X,XI))/
     1(AMU+DELMU/3.-A)
      YY2=(Q2YY(Y,ETA,X,XI,AMU+2.*DELMU/3.)-Q2OYY(Y,ETA,X,
     1XI))/(AMU+2.*DELMU/3.-A)
      YY3=(Q2YY(Y,ETA,X,XI,AMU+DELMU)-Q2OYY(Y,ETA,X,XI))/
     1(AMU+DELMU-A)
      GO TO 325
  320 CONTINUE
      YY0=(Q2YY(Y,ETA,X,XI,AMU+DELMU)-Q2OYY(Y,ETA,X,XI))/
     1(AMU+DELMU-A)
      GO TO 350
  325 CONTINUE
      SUMYY=SUMYY+(DELMU/8.)*(YY0+3.*YY1+3.*YY2+YY3)
      YY0=YY3
  350 AMU=AMU+DELMU
      PV2FYY=2.*SUMYY

    EVALUATION OF THE INFINITE INTEGRAL IN G, GN, AND GYY
      SIMULTANEOUSLY

      AMU=2*A
      DELMU0=DELMU
      F0=Q1(Y,ETA,XX,AMU)/(AMU-A)
      F0X=Q2X(Y,ETA,X,XI,AMU)/(AMU-A)
      F0Y=Q2Y(Y,ETA,X,XI,AMU)/(AMU-A)
      F0YY=Q2YY(Y,ETA,X,XI,AMU)/(AMU-A)
      SUM=0.0
      SUMX=0.0
      SUMY=0.0
      SUMYY=0.0
      DO 100 NN=1,200
      DO 95 N=1,20
      F1=Q1(Y,ETA,XX,AMU+DELMU/3.)/(AMU+DELMU/3.-A)
      F1X=Q2X(Y,ETA,X,XI,AMU+DELMU/3.)/(AMU+DELMU/3.-A)
      F1Y=Q2Y(Y,ETA,X,XI,AMU+DELMU/3.)/(AMU+DELMU/3.-A)
      F1YY=Q2YY(Y,ETA,X,XI,AMU+DELMU/3.)/(AMU+DELMU/3.-A)
      F2=Q1(Y,ETA,XX,AMU+2.*DELMU/3.)/(AMU+2.*DELMU/3.-A)
```

```
      F2X=Q2X(Y,ETA,X,XI,AMU+2.*DELMU/3.)/(AMU+2.*DELMU/3.
     1-A)
      F2Y=Q2Y(Y,ETA,X,XI,AMU+2.*DELMU/3.)/(AMU+2.*DELMU/3.
     1-A)
      F2YY=Q2YY(Y,ETA,X,XI,AMU+2.*DELMU/3.)/(AMU+2.*DELMU/3.
     1-A)
      F3=Q1(Y,ETA,XX,AMU+    DELMU)/(AMU+DELMU-A)
      F3X=Q2X(Y,ETA,X,XI,AMU+DELMU)/(AMU+DELMU-A)
      F3Y=Q2Y(Y,ETA,X,XI,AMU+DELMU)/(AMU+DELMU-A)
      F3YY=Q2YY(Y,ETA,X,XI,AMU+DELMU)/(AMU+DELMU-A)
      SUM=    (DELMU/8.)*(F0+3.*F1+3.*F2+F3)+SUM
      SUMX=    (DELMU/8.)*(F0X+3.*F1X+3.*F2X+F3X)+SUMX
      SUMY=    (DELMU/8.)*(F0Y+3.*F1Y+3.*F2Y+F3Y)+SUMY
      SUMYY=SUMYY+(DELMU/8.)*(F0YY+3.*F1YY+3.*F2YY+F3YY)
      F0=F3
      F0X=F3X
      F0Y=F3Y
      F0YY=F3YY
      ASM=EXP(AMU*(ETA+Y))/AMU
      IF(ASM.LT..0001) GO TO 86
   95 AMU=AMU+DELMU
   86 TEST(NN)=SUM
      TESTT(NN)=SUMX
      TESTTT(NN)=SUMY
      TESTYY(NN)=SUMYY
      IF(NN-1) 100,100,97
   97 CONTINUE
      IF(ASM.LT..00010) GO TO 105
      IF(ABS(SUM).LT..0001) GO TO 98
      IF(ABS((TEST(NN)-TEST(NN-1))/TEST(NN)).GT..001)
     1GO TO 100
   98 IF(ABS(SUMX).LT..0001) GO TO 99
      IF(ABS((TESTT(NN)-TESTT(NN-1))/TESTT(NN)).GT..001)
     1GO TO 100
   99 IF(ABS(SUMY).LT..0001) GO TO 93
      IF(ABS((TESTTT(NN)-TESTTT(NN-1))/TESTTT(NN)).GT..001)
     1GO TO 100
   93 IF(ABS(SUMYY).LT.0.0001) GO TO 96
      IF (ABS((TESTYY(NN)-TESTYY(NN-1))/TESTYY(NN)).GT..001)
     1GO TO 100
   96 CONTINUE
      GO TO 105
  100 CONTINUE
  102 WRITE(6,103)
  103 FORMAT(3X34HINFINITE INTEGRAL DID NOT CONVERGE)
  105 CONTINUE
      PV1I=2.*SUM
      PV2IX=2.*SUMX
      PV2IY=2.*SUMY
      PV2IYY=2.*SUMYY
      DELMU=.2/H
      IF (J.EQ.25) GO TO 240
      AMU=0.0
      SUM=0.0
      SUMX=0.0
      SUMY=0.0
      SUMYY=0.0
      F0=0.0
      F0X=FUN3X(Y,ETA,X,XI,AMU)
      F0Y=FUN3Y(Y,ETA,X,XI,AMU)
      F0YY=FUN3YY(Y,ETA,X,XI,AMU)
      DO 200 NN=1,100
      IF(AMU*H.GT.5.) DELMU=.1
      DO 195 N=1,20
      F1X=FUN3X(Y,ETA,X,XI,AMU+DELMU/3.)
      F2X=FUN3X(Y,ETA,X,XI,AMU+2.*DELMU/3.)
      F3X=FUN3X(Y,ETA,X,XI,AMU+DELMU)
      F1Y=FUN3Y(Y,ETA,X,XI,AMU+DELMU/3.)
      F2Y=FUN3Y(Y,ETA,X,XI,AMU+2.*DELMU/3.)
      F3Y=FUN3Y(Y,ETA,X,XI,AMU+DELMU)
      F1YY=FUN3YY(Y,ETA,X,XI,AMU+DELMU/3.)
      F2YY=FUN3YY(Y,ETA,X,XI,AMU+2.*DELMU/3.)
```

```
      F3YY=FUN3YY(Y,ETA,X,XI,AMU+DELMU)
      IF(AMU+DELMU.LT..001) GO TO 120
      F1=FUN1(Y,ETA,XX,AMU+DELMU/3.)
      F2=FUN1(Y,ETA,XX,AMU+2.*DELMU/3.)
      F3=FUN1(Y,ETA,XX,AMU+DELMU)
      GO TO 130
  120 CONTINUE
      F1=EXP(-(AMU+DELMU/3.)*H)*COS((AMU+DELMU/3.)*XX)*
     1(AMU+DELMU/3.)*Y*ETA/COSH((AMU+DELMU/3.)*H)
      F2=EXP(-(AMU+2.*DELMU/3.)*H)*COS((AMU+2.*DELMU/3.)*XX)
     1*(AMU+2.*DELMU/3.)*Y*ETA/COSH((AMU+2.*DELMU/3.)*H)
      F3=EXP(-(AMU+DELMU)*H)*COS((AMU+DELMU)*XX)*(AMU+DELMU)
     1*Y*ETA/COSH((AMU+DELMU)*H)
  130 CONTINUE
      SUM=    (DELMU/8.)*(F0+3.*F1+3.*F2+F3)+SUM
      SUMX=    (DELMU/8.)*(F0X+3.*F1X+3.*F2X+F3X)+SUMX
      SUMY=    (DELMU/8.)*(F0Y+3.*F1Y+3.*F2Y+F3Y)+SUMY
      SUMYY=SUMYY+(DELMU/8.)*(F0YY+3.*F1YY+3.*F2YY+F3YY)
      F0=F3
      F0X=F3X
      F0Y=F3Y
      F0YY=F3YY
      ASM=EXP(AMU*(ETA+Y))
      IF(ASM.LT..0001) GO TO 196
  195 AMU=AMU+DELMU
  196 TEST(NN)=SUM
      TESTT(NN)=SUMX
      TESTTT(NN)=SUMY
      TESTYY(NN)=SUMYY
      IF(NN-1) 200,200,199
  199 CONTINUE
      IF(ASM.LT..00010) GO TO 205
      IF(ABS(SUM).LT..0001) GO TO 206
      IF(ABS((TEST(NN)-TEST(NN-1))/TEST(NN)).GT..0010)
     1GO TO 200
  206 IF(ABS(SUMX).LT..0001) GO TO 207
      IF(ABS((TESTT(NN)-TESTT(NN-1))/TESTT(NN)).GT..0010)
     1GO TO 200
  207 IF(ABS(SUMY).LT..0001) GO TO 204
      IF(ABS((TESTTT(NN)-TESTTT(NN-1))/TESTTT(NN)).GT..0010)
     1GO TO 200
  204 IF(ABS(SUMYY).LT.0.0001) GO TO 208
      IF (ABS((TESTYY(NN)-TESTYY(NN-1))/(TESTYY(NN))).
     1GT..001) GO TO 200
  208 CONTINUE
      GO TO 205
  200 CONTINUE
      WRITE(6,202)
  202 FORMAT(3X14HNO CONVERGENCE)
  205 CONTINUE
      GINF=-2*SUM
      GXINF=2.*SUMX
      GYINF=2.*SUMY
      GYYINF=2.*SUMYY
      GNSING=.5
      IF (I.EQ.25) GO TO 218
      IF(I.EQ.J) GO TO 220
      AIJJ=I-J
      THETA1=ABS(AIJJ)*DELTHE
      AINJJ=I+NPTS-J
      THETA2=ABS(AINJJ)*DELTHE
      AJNII=I-J-NPTS
      THETA3=ABS(AJNII)*DELTHE
      THETA=THETA1
      IF(THETA2.LT.THETA) THETA=THETA2
      IF(THETA3.LT.THETA) THETA=THETA3
      IF(THETA.GT..15) GO TO 218
      GSING=DELTHE*ALOG(2.)+2.*(-DELTHE/2.+(DELTHE/4.+
     1THETA/2.)*ALOG(THETA/2.+DELTHE/4.)-(THETA/2.-DELTHE/4.
     2)*ALOG(THETA/2.-DELTHE/4.)-(DELTHE/4.+THETA/2.)**3/18.
     3+(THETA/2.-DELTHE/4.)**3/18.-(THETA/2.+DELTHE/4.)**5/
     4(180.*5.)+(THETA/2.-DELTHE/4.)**5/(180.*5.)-(THETA/2.+
```

```
5DELTHE/4.)**7/(2835.*7.)+(THETA/2.-DELTHE/4.)**7/
6(2835.*7.))
 GXSING=(X-XI)/((X-XI)**2+(Y-ETA)**2)
 GYSING=(Y-ETA)/((X-XI)**2+(Y-ETA)**2)
 GO TO 230
218 GSING=DELTHE*ALOG(SQRT((X-XI)**2+(Y-ETA)**2))
 GXSING=(X-XI)/((X-XI)**2+(Y-ETA)**2)
 GYSING=(Y-ETA)/((X-XI)**2+(Y-ETA)**2)
 GO TO 230
220 CONTINUE
 GSING=DELTHE*ALOG(DELTHE/2.)-DELTHE-(DELTHE**3)/
1(18.*16.)-(DELTHE**5)/(180.*5.*256.)-(DELTHE**7)/
2(2835.*7.*256.*16.)
 GXSING=GNSING*ANX
 GYSING=GNSING*ANY
230 CONTINUE
 G  =GSING/DELTHE-ALOG(SQRT((X-XI)**2+(Y+ETA)**2))+PV1F
1+PV1I+GINF+CMPLX(0.,-1.)*FUN2(Y,ETA,XX)
 GX=GXSING-FUNR(X,Y,XI,ETA,1.0,0.0)+PV2FX+PV2IX+GXINF+
1FUN4(Y,ETA,XX,1.0,0.0,SIGN)*CMPLX(0.0,-1.0)
 GY=GYSING-FUNR(X,Y,XI,ETA,0.0,1.0)+PV2FY+PV2IY+GYINF+
1FUN4(Y,ETA,XX,0.0,1.0,SIGN)*CMPLX(0.0,-1.0)
 IF (I.LE.NPTS) GO TO 235
 FURYY=1./SQRT(((X-XI)**2+(Y-ETA)**2)**3)-((Y-ETA)**2+
1(Y-ETA))/SQRT(((X-XI)**2+(Y-ETA)**2)**5)
 FUNYY=1./SQRT(((X-XI)**2+(Y+ETA)**2)**3)-((Y+ETA)**2+
1(Y+ETA))/SQRT(((X-XI)**2+(Y+ETA)**2)**5)
 GYY=FURYY+FUNYY+PV2FYY+PV2IYY+GYYINF+CMPLX(0.0,-1.0)*
1FUN2(Y,ETA,XX)*A*A
235 CONTINUE
 IF (I.EQ.25) GO TO 250
 GN=GNSING-FUNR(X,Y,XI,ETA,ANX,ANY)+(PV2FX+PV2IX+GXINF)
1*ANX+(PV2FY+PV2IY+GYINF)*ANY+FUN4(Y,ETA,XX,ANX,ANY,
2SIGN)*CMPLX(0.0,-1.0)
 GO TO 250
240 CONTINUE
 G=-(PV1F+PV1I)+CMPLX(0.,-1.)*FUN2(Y,ETA,XX)
 GN=-(PV2FX+PV2IX)*ANX-(PV2FY+PV2IY)*ANY+FUN4(Y,ETA,XX,
1ANX,ANY,SIGN)*CMPLX(0.0,-1.0)
250 CONTINUE
 RETURN
 END


 THIS SUBROUTINE CALCULATES G AND ITS DERIVATIVES,
 GX,GY,GN, AND GYY, BY USE OF THE SERIES FORM FOR THE
 CASE (X(I) - X(J)) GREATER THAN SMIN


 SUBROUTINE GREENS (A,ANU,SH2AH,SHAH,CHAH,COSAMU,SINAMU
1,AMU,COEFG,SHYI,CHYI,SHYJ,CHYJ,I,J,XV,YV,XC,YC,ANXI,
2ANYI,ANXJ,ANYJ,GIJ,GNIJ,GNJI,GXIJ,GXJI,GYIJ,GYJI,GYY)

 COMPLEX GIJ,GNIJ,GNJI,GXIJ,GXJI,GYIJ,GYJI,GYY
 DIMENSION COSAMU(200,25),SINAMU(200,25),AMU(200),
1COEFG(200)
 DIMENSION TEST1(40),TEST2(40),TEST3(40),TEST30(40),
1TEST4(40)
 COMMON/CONST/H,D,DELTHE,SMIN,NPTS,NSPTS
 COMMON/CPI/PI
 SUM1=0.0
 SUM2=0.0
 SUM3=0.0
 SUM30=0.0
 SUM4=0.0
 DO 20 N=1,40
 DO 15 KK=1,5
 K=(N-1)*5+KK
 SNI=SINAMU(K,I)
 SNJ=SINAMU(K,J)
 CSI=COSAMU(K,I)
 CSJ=COSAMU(K,J)
```

```
      AK=AMU(K)
      COK=COEFG(K)
      VAL=AK*ABS(XV-XC)
      IF (VAL.GE.75.0) GO TO 10
      EXIJ=EXP(-VAL)
      GO TO 11
   10 EXIJ=0.0
   11 CONTINUE
      SUM1=SUM1+COK*CSI*CSJ*EXIJ
      SUM2=SUM2+COK*CSI*CSJ*EXIJ*AK
      SUM3=SUM3+COK*AK*SNI*CSJ*EXIJ
      SUM30=SUM30+COK*AK*SNJ*CSI*EXIJ
      SUM4=SUM4+AK*AK*COK*CSI*CSJ*EXIJ
   15 CONTINUE
      TEST1(N)=SUM1
      TEST2(N)=SUM2
      TEST3(N)=SUM3
      TEST30(N)=SUM30
      TEST4(N)=SUM4
      IF(N-1) 20,20,16
   16 IF(ABS(SUM1).LE.0.000001) GO TO 17
      IF(ABS((TEST1(N)-TEST1(N-1))/TEST1(N)).GT..0010) GO
     1TO 20
   17 IF(ABS(SUM30).LE.0.000001) GO TO 18
      IF(ABS((TEST30(N)-TEST30(N-1))/TEST30(N)).GT..0010)
     1GO TO 20
   18 IF(ABS(SUM3).LE.0.000001) GO TO 19
      IF(ABS((TEST3(N)-TEST3(N-1))/TEST3(N)).GT..0010) GO
     1TO 20
   19 IF(ABS(SUM2).LE.0.000001) GO TO 21
      IF(ABS((TEST2(N)-TEST2(N-1))/TEST2(N)).GT..0010) GO
     1TO 20
   21 IF(ABS(SUM4).LE.0.000001) GO TO 30
      IF(ABS((TEST4(N)-TEST4(N-1))/TEST4(N)).GT..0010) GO
     1TO 20
      GO TO 30
   20 CONTINUE
      WRITE(6,25) I,J
   25 FORMAT(10X23HGREENS DID NOT CONVERGE,2X2HI=I2,
     12X2HJ=I2)
   30 CONTINUE
      IF (XV.LT.XC) SIGN=-1.0
      IF (XV.GT.XC) SIGN=1.0
      IF (XV.EQ.XC) SIGN=0.0
      TERM4=4.*PI*CHYI*CHYJ*SIN(A*ABS(XV-XC))/(2.*A*H+SH2AH)
      TERM5=4.*PI*CHYI*CHYJ*COS(A*ABS(XV-XC))/(2.*A*H+SH2AH)
      TERM6=A*TERM5*SIGN
      TERM7=A*TERM4*SIGN
      TERM8=4.*PI*A*SHYI*CHYJ*COS(A*ABS(XV-XC))/(2.*A*H+
     1SH2AH)
      TERM9=4.*PI*A*SHYI*CHYJ*SIN(A*ABS(XV-XC))/(2.*A*H+
     1SH2AH)
      TERM80=4.*PI*A*SHYJ*CHYI*COS(A*ABS(XV-XC))/(2.*A*H+
     1SH2AH)
      TERM90=4.*PI*A*SHYJ*CHYI*SIN(A*ABS(XV-XC))/(2.*A*H+
     1SH2AH)
      TERM10=A*A*TERM4
      TERM11=A*A*TERM5
      IF (J.EQ.25) GO TO 70
      GIJ=CMPLX(SUM1+TERM4,-TERM5)
      GXIJ= CMPLX(-SIGN*SUM2+TERM6,TERM7)
      GYIJ=CMPLX(-SUM3+TERM9,-TERM8)
      IF (I.EQ.25) GO TO 40
      GXJI=CMPLX(SIGN*SUM2-TERM6,-TERM7)
      GYJI=CMPLX(-SUM30+TERM90,-TERM80)
      GNIJ=CMPLX(-ANXI*SIGN*SUM2+ANXI*TERM6-ANYI*SUM3+ANYI*
     1TERM9,ANXI*TERM7-ANYI*TERM8)
      GNJI=CMPLX(ANXJ*SIGN*SUM2-ANXJ*TERM6-ANYJ*SUM30+ANYJ*
     1TERM90,-ANXJ*TERM7-ANYJ*TERM80)
      GO TO 60
   40 CONTINUE
      GYY=CMPLX(-SUM4+TERM10,-TERM11)
```

```
60 CONTINUE
   GO TO 80
70 GIJ=CMPLX(-SUM1-TERM4,-TERM5)
   GNIJ=CMPLX(ANXI*SIGN*SUM2-ANXI*TERM6+ANYI*SUM3-ANYI*
  1TERM9,ANXI*TERM7-ANYI*TERM8)
80 CONTINUE
   RETURN
   END


   THIS SUBROUTINE INVERTS COMPLEX MATRICES TO SOLVE
   THE MATRIX EQUATION ALPHA(I,J)*F(I,1) = HH(I,1) AND
   ALPHA(I,J)*F1(I,1) = PK(I,1)


   SUBROUTINE COMAT(N,M,A,B,D,I)

   INTEGER C,H,R,Q,Z
   COMPLEX A,B,D,TT,P
   DIMENSION A(N,N),B(N,M),C(100,3)
   D = (1.0,0.0)
   DO 20 J = 1,N
20 C(J,3) = 0
   DO 21 K = 1,N
   TT = (0.0,0.0)
   T = 0.0
   DO 4 J = 1,N
   IF (C(J,3) .EQ. 1) GO TO 4
   DO 5 H = 1,N
   IF (C(H,3) - 1) 15,5,12
15 IF (T .GE.      CABS(A(J,H)))  GO TO 5
   R = J
   Q = H
   T =       CABS(A(J,H))
 5 CONTINUE
 4 CONTINUE
   C(Q,3) = C(Q,3) + 1
   C(K,1) = R
   C(K,2) = Q
   IF (R .EQ. Q) GO TO 11
   D = -D
   DO 8 L = 1,N
   TT = A(R,L)
   A(R,L) = A(Q,L)
 8 A(Q,L) = TT
   IF(M .LE. 0) GO TO 11
   DO 2 L = 1,M
   TT = B(R,L)
   B(R,L) = B(Q,L)
 2 B(Q,L) = TT
11 P = A(Q,Q)
   A(Q,Q) = (1.0,0.0)
   DO 13 L = 1,N
13 A(Q,L) = A(Q,L)/P
   IF (M .LE. 0) GO TO 1
   DO 3 L = 1,M
 3 B(Q,L) = B(Q,L)/P
 1 DO 21 Z = 1,N
   IF (Z .EQ. Q) GO TO 21
   TT = A(Z,Q)
   A(Z,Q)=(0.0,0.0)
   DO 16 L = 1,N
16 A(Z,L) = A(Z,L) - A(Q,L)*TT
   IF (M .LE. 0) GO TO 21
   DO 17 L = 1,M
17 B(Z,L) = B(Z,L) - B(Q,L)*TT
21 CONTINUE
   DO 19 II = 1,N
   L = N + 1 - II
   IF(C(L,1) .EQ. C(L,2)) GO TO 19
   R = C(L,1)
   Q = C(L,2)
```

```
      DO 7 K = 1,N
      TT = A(K,R)
      A(K,R) = A(K,Q)
    7 A(K,Q) = TT
   19 CONTINUE
      DO 18 K = 1,N
      IF (C(K,3) .NE. 1) GO TO 12
   18 CONTINUE
      I = 1
   50 RETURN
   12 I = 2
      GO TO 50
      END

//GO.SYSIN   DD *
```

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